1. Interpretations.

1.1. Review of terms and statements.

Remark 1.1. Suppose t is a term. Then exactly one of the following hold:

(T1) $t = x$ for some $x \in X$;
(T2) $t = c$ for some $c \in C$;
(T3) $t = f(t_1, \ldots, t_n)$ for some $n \in \mathbb{N}^+$, some $f \in F_n$ and some terms $t_1, \ldots, t_n$.

Suppose A is a statement. Then exactly one of the following hold:

(S1) $A = r(t_1, \ldots, t_n)$ for some $n \in \mathbb{N}^+$, some $r \in R_n$ and some terms $t_1, \ldots, t_n$;
(S2) $A = \forall x\, B$ for some $x \in X$ and some statement B;
(S3) $A = \exists x\, B$ for some $x \in X$ and some statement B;
(S4) $A = (\ s\ =\ t\ )$ for some terms s, t;
(S5) $A = \sim B$ for some statement B;
(S6) $A = (\ B\, b\, C\ )$ for some statements B, C and some $b \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$.

1.2. Interpretations.

Definition 1.1. An **interpretation** of a first order language is an ordered quadruple

$$\mathcal{I} = (D, \mathbf{C}, \mathbf{F}, \mathbf{R})$$

such that D is a set and $\mathbf{C}$, $\mathbf{F}$, $\mathbf{R}$ are functions with domains C, F, R, respectively, such that

$$\mathbf{rng}\, \mathbf{C} \subset D;$$

for each $n \in \mathbb{N}^+$ and each $f \in F_n$,

$$\mathbf{F}(f) : D^n \rightarrow D;$$

for each $n \in \mathbb{N}^+$ and each $r \in R_n$,

$$\mathbf{R}(r) : D^n \rightarrow \{0, 1\};$$

Let us fix and interpretation

$$\mathcal{I} = (D, \mathbf{C}, \mathbf{F}, \mathbf{R}).$$

Remark 1.2. In Hodel an **assignment** is a function $\alpha : X \rightarrow D$ which is to say

$$\alpha \in D^X.$$

Definition 1.2. If $\alpha, \beta \in D^X$ and $x \in X$ we write

$$\alpha \sim_x \beta \quad \text{if } \alpha(y) = \beta(y) \text{ whenever } y \in X \sim \{x\}.$$

Proposition 1.1. Suppose $\alpha \in D^X$. There is one and only one way to assign each term t a member

$$t_\alpha \quad \text{of } D$$

such that

(i) $t_\alpha = \alpha(x)$ if (T1) holds;
(ii) $t_\alpha = c$ if (T2) holds;
(iii) $t_\alpha = \mathbf{F}(f)((t_1)_\alpha, \ldots, (t_n)_\alpha)$ if (T3) holds.

Moreover, if $\alpha, \beta \in D^X$ and

$$\alpha(x) = \beta(x) \quad \text{whenever } x \in X \text{ and } x \text{ occurs in } t$$

then

$$t_\alpha = t_\beta.$$

Proposition 1.2. Induct on the depth of a parse tree for t.

Proposition 1.3. Suppose A is a statement. There is one and only one function

$$\mathbf{t}_A : D^X \to \{0,1\}$$

such that

(i) if (S1) holds then

$$\mathbf{t}_A(\alpha) = \mathbf{R}(r)((t_1)_\alpha, \ldots, (t_n)_\alpha)$$

for $\alpha \in D^X$;

(ii) if (S2) holds then

$$\mathbf{t}_A(\alpha) = \begin{cases} 1 & \text{if } \mathbf{t}_B(\beta) = 1 \text{ for all assignments } \beta \text{ such that } \alpha \sim_x \beta, \\ 0 & \text{if } \mathbf{t}_B(\beta) = 0 \text{ for some assignment } \beta \text{ such that } \alpha \sim_x \beta \end{cases}$$

for $\alpha \in D^X$;

(iii) if (S3) holds then

$$\mathbf{t}_A(\alpha) = \begin{cases} 1 & \text{if } \mathbf{t}_B(\beta) = 1 \text{ for some assignment } \beta \text{ such that } \alpha \sim_x \beta, \\ 0 & \text{if } \mathbf{t}_B(\beta) = 0 \text{ for all assignments } \beta \text{ such that } \alpha \sim_x \beta \end{cases}$$

for $\alpha \in D^X$;

(iv) if (S4) holds then

$$\mathbf{t}_A(\alpha) = \begin{cases} 1 & \text{if } s_\alpha = t_\alpha, \\ 0 & \text{if } s_\alpha \neq t_\alpha \end{cases}$$

for $\alpha \in D^X$;

(v) if (S5) holds $\mathbf{t}_A = \sim \mathbf{t}_B$;

(v) if (S6) holds then

$$\mathbf{t}_A = \mathbf{t}_B \, b \, \mathbf{t}_C;$$

Proof. Induct on the depth of a parse tree for A. □

Definition 1.3. (Tarski's definition of truth.) We say the statement A is **true in** $\mathcal{I}$ if

$$\mathbf{t}_A(\alpha) = 1 \quad \text{for all } \alpha \in X^D.$$

We say the statement A is **false in** $\mathcal{I}$ if

$$\mathbf{t}_A(\alpha) = 0 \quad \text{for all } \alpha \in X^D.$$

So a statement may be neither true nor false in $\mathcal{I}$.

Definition 1.4. We say the statement A is **logically valid** if it is true in every interpretation. We say two statements A and B are **logically equivalent** if $A \leftrightarrow B$ is logically valid.

Proposition 1.4. Suppose B is a statement. Then

$$\exists x \, B \quad \text{and} \quad \sim \forall x \sim B$$

are logically equivalent as are

$$\forall x \, B \quad \text{and} \quad \sim \exists x \sim B.$$

Proposition 1.5. Suppose A is a statement and x is a variable. Then

$$\forall x\, A \rightarrow A$$

is logically valid.

Proof. Let $B = \forall x\, A$. Suppose $\alpha \in D^X$ and $\mathbf{t}_B(\alpha) = 1$. Then $\mathbf{t}_A(\alpha) = 1$ since $\alpha \sim_x \alpha$. Thus $\mathbf{t}_{B\rightarrow A}(\alpha) = 1$. □

Proposition 1.6. (Modus ponens for interpretations.) Suppose A and B are statements and A and $A \rightarrow B$ are true in $\mathcal{I}$. Then B is true in $\mathcal{I}$.

Proof. Suppose $\alpha \in D^X$,

$$\mathbf{t}_A(\alpha) = 1 \quad \text{and} \quad \mathbf{t}_{A\rightarrow B}(\alpha) = 1.$$

Since

$$\mathbf{t}_{A\rightarrow B} = \sim \mathbf{t}_A \vee \mathbf{t}_B$$

we infer that $\mathbf{t}_B(\alpha) = 1$. □

Theorem 1.1. Suppose A is a statement, $\alpha, \beta \in D^X$ and

$$\alpha(x) = \beta(x) \quad \text{for } x \in \mathbf{free}(A).$$

Then

$$\mathbf{t}_A(\alpha) = \mathbf{t}_A(\beta).$$

Proof. **Part One.** Suppose (S1) holds. From a previous theorem we have $(t_i)_\alpha = (t_i)_\beta$ for $i = 1, \ldots, n$. Thus

$$\mathbf{t}_A(\alpha) = \mathbf{R}(r)((t_1)_\alpha, \ldots, (t_n)_\alpha) = \mathbf{R}(r)((t_1)_\beta, \ldots, (t_n)_\beta) = \mathbf{t}_A(\beta).$$

Part Two. Suppose (S2) holds. Let us assume inductively that

(1) $\quad \mathbf{t}_B(\gamma) = \mathbf{t}_B(\delta)$ whenever $\gamma, \delta \in D^X$ and $\gamma(x) = \delta(x)$ for $x \in \mathbf{free}(B)$.

Suppose $\mathbf{t}_A(\alpha) = 1$. Then

(2) $\quad \mathbf{t}_B(\gamma) = 1$ whenever $\gamma \in D^X$ and $\gamma \sim_x \alpha$.

Suppose $\gamma \in D^X$ and $\gamma \sim_x \beta$. Let $\zeta \in D^X$ be such that

$$\zeta(y) = \begin{cases} \alpha(y) & \text{for } y \in X \sim \{x\}, \\ \gamma(x) & \text{for } y = x. \end{cases}$$

Since γ and ζ agree on $\mathbf{free}(B) = \mathbf{free}(A) \sim \{x\}$ we find that $\mathbf{t}_B(\gamma) = \mathbf{t}_B(\zeta)$. Since $\zeta \sim_x \alpha$ and $\mathbf{t}_A(\alpha) = 1$ we infer that $\mathbf{t}_B(\zeta) = 1$. Thus $\mathbf{t}_B(\gamma) = 1$ which implies $\mathbf{t}_A(\beta) = 1$.

Interchanging α and β we find that if $\mathbf{t}_A(\beta) = 1$ then $\mathbf{t}_A(\alpha) = 1$. Thus $\mathbf{t}_A(\alpha) = \mathbf{t}_A(\beta)$.

Part Three. Suppose (S3) holds. Using the result of Part Two and noting that

$$\mathbf{free}(\forall x \sim B) = \mathbf{free}(A)$$

we find that

$$\mathbf{t}_A(\alpha) = \sim \mathbf{t}_{\forall x \sim B}(\alpha) = \sim \mathbf{t}_{\forall x \sim B}(\beta) = \mathbf{t}_A(\beta).$$

Part Four. Suppose (S4) holds. Then $\mathbf{free}(A) = \mathbf{free}(s) \cup \mathbf{free}(t)$ so $s_\alpha = s_\beta$ and $t_\alpha = t_\beta$. It follows immediately that $\mathbf{t}_A(\alpha) = \mathbf{t}_A(\beta)$.

Part Five. Suppose (S5) holds. Then $\mathbf{free}(A) = \mathbf{free}(B)$ so we may assume inductively that $\mathbf{t}_B(\alpha) = \mathbf{t}_B(\beta)$. Thus

$$\mathbf{t}_A(\alpha) = \sim \mathbf{t}_B(\alpha) = \sim \mathbf{t}_B(\beta) = \mathbf{t}_A(\beta).$$

Part Six. Suppose (S6) holds. Then $\mathbf{free}(A) = \mathbf{free}(B) \cup \mathbf{free}(C)$ so we may assume inductively that $\mathbf{t}_B(\alpha) = \mathbf{t}_B(\beta)$ and $\mathbf{t}_C(\alpha) = \mathbf{t}_C(\beta)$. Thus

$$\mathbf{t}_A(\alpha) = \mathbf{t}_B(\alpha)\, b\, \mathbf{t}_C(\alpha)\mathbf{t}_B(\beta)\, b\, \mathbf{t}_C(\beta) = \mathbf{t}_A(\beta).$$

□

Corollary 1.1. Suppose A is a statement, $\alpha, \beta \in D^X$ and

$$\alpha(x) = \beta(x) \quad \text{for } x \in \mathbf{free}(A).$$

Then *either*

$$\mathbf{t}_A(\gamma) = 1 \quad \text{for } \gamma \in D^X$$

or

$$\mathbf{t}_A(\gamma) = 0 \quad \text{for } \gamma \in D^X.$$

2. Substitution.

Definition 2.1. Suppose t is a term and $x \in X$.
If s is a term then

$$s_{x\to t}$$

is the string obtained by replacing each occurrence of x in s by t.
If A is a statement then

$$A_{x\to t}$$

is the string obtained by replacing each *free* occurrence of x in A by t.

Proposition 2.1. Suppose t is a term and $x \in X$.
If s is a term then $s_{x\to t}$ is a term.
If A is a statement then $A_{x\to t}$ is a statement.

Proof. Just consider parse trees. □

Remark 2.1. $A_{x\to t}$ is not very useful unless t is free for x in A.

Now let us fix an interpretation

$$\mathcal{I} = (D, \mathbf{C}, \mathbf{F}, \mathbf{R}).$$

Proposition 2.2. Suppose s, t are terms, $x \in X$ and $\alpha, \beta \in D^X$ are such that

$$\alpha \sim_x \beta \quad \text{and} \quad t_\alpha = x_\beta.$$

Then

$$(s_{x\to t})_\alpha = s_\beta.$$

Proof. We induct on the depth of a parse tree for s.
Suppose $s = x$. Then

$$(s_{x\to t})_\alpha = t_\alpha = x_\beta = \beta(x) = s_\beta.$$

Suppose $y \in X \sim \{x\}$ and $s = y$. Then

$$(s_{x\to t})_\alpha = y_\alpha = \alpha(y) = \beta(y) = s_\beta.$$

Suppose $n \in \mathbb{N}^+$, $f \in F_n$, $t_1, \ldots, t_n$ are terms and $s = f(t_1, \ldots, t_n)$. By the inductive hypothesis we have

$$\big((t_i)_{x\to t}\big)_\alpha = (t_i)_\beta \quad i = 1, \ldots, n.$$

Thus

$$\begin{aligned}(s_{x\to t})_\alpha &= \big(s((t_1)_{x\to t}, \ldots, (t_n)_{x\to t})\big)_\alpha \\ &= \mathbf{F}(f)(((t_1)_{x\to t})_\alpha, \ldots, ((t_n)_{x\to t})_\alpha) \\ &= \mathbf{F}(f)((t_1)_\beta, \ldots, (t_n)_\beta) \\ &= (s(t_1, \ldots, t_n))_\beta.\end{aligned}$$

□

Theorem 2.1. Suppose A is a statement, t is a term, $x \in X$, x is free for t in A and $\alpha, \beta \in D^X$ are such that

$$\alpha \sim_x \beta \quad \text{and} \quad t_\alpha = x_\beta.$$

Then

$$\mathbf{t}_{A_{x\to t}}(\alpha) = \mathbf{t}_A(\beta).$$

Proof. **Part One.** Suppose (S1) holds. Then

$$A_{x\to t} = r((t_1)_{x\to t}, \ldots, (t_n)_{x\to t})$$

so

$$\begin{aligned}t_{A_{x\to t}}(\alpha) &= \mathbf{R}(r)(((t_1)_{x\to t})_\alpha, \ldots, ((t_n)_{x\to t})_\alpha) \\ &= \mathbf{R}(r)((t_1)_\beta, \ldots, (t_n)_\beta) \\ &= \mathbf{t}_A(\beta).\end{aligned}$$

Part Two. Suppose (S2) holds with x there equal $w \in X$; that is, $A = \forall w\, B$ and t is free for x in A. Let $Y = \{y \in X : y \text{ occurs in } t\}$.

Suppose there are no free occurrences of x in A. Then $A_{x\to t} = A$ and, as α agrees with β on $\mathbf{free}(A)$, we have $\mathbf{t}_A(\alpha) = \mathbf{t}_A(\beta)$ so the Theorem holds in this case.

Suppose there are free occurrences of x in A. Then $w \neq x$ so

$$A_{x\to t} = \forall w\, B_{x\to t}.$$

Since t is free for x in A we find that $w \notin Y$ which implies that t is free for x in B.

Suppose $\mathbf{t}_{\forall w\, B_{x\to t}}(\alpha) = 1$. Then

$$\mathbf{t}_{B_{x\to t}}(\gamma) = 1 \quad \text{whenever } \gamma \in D^X \text{ and } \gamma \sim_w \alpha.$$

Suppose $\delta \in D^X$ and $\delta \sim_w \beta$. Let

$$\gamma(y) = \begin{cases}\alpha(y) & \text{if } y \neq w, \\ \delta(y) & \text{if } y = w.\end{cases}$$

Since $\gamma \sim_w \alpha$ we have $t_{B_{x\to t}}(\gamma) = 1$. Since $t_\gamma = t_\alpha$ and $x_\delta = x_\beta$ we may assume inductively that

$$\mathbf{t}_{B_{x\to t}}(\gamma) = \mathbf{t}_B(\delta)$$

so that $\mathbf{t}_B(\delta) = 1$. Thus $\mathbf{t}_{\forall w\, B}(\beta) = 1$.

Suppose $\mathbf{t}_{\forall w\, B}(\beta) = 1$. Then

$$\mathbf{t}_B(\delta) = 1 \quad \text{whenever } \delta \in D^X \text{ and } \delta \sim_w \beta.$$

Suppose $\gamma \in D^X$ and $\gamma \sim_w \alpha$. Let

$$\delta(y) = \begin{cases} \beta(y) & \text{if } y \neq w, \\ \gamma(y) & \text{if } y = w. \end{cases}$$

Since $\delta \sim_w \beta$ we have $t_B(\delta) = 1$. Since $t_\gamma = t_\alpha$ and $x_\delta = x_\beta$ we may assume inductively that

$$\mathbf{t}_{B_{x\to t}}(\gamma) = \mathbf{t}_B(\delta)$$

so that $\mathbf{t}_{B_{x\to t}}(\alpha) = 1$. Thus $\mathbf{t}_{\forall w\, B_{x\to t}}(\alpha) = 1$.

Part Three. Suppose (S3) holds with x there equal $w \in X$; that is, $A = \exists w\, B$ and t is free for x in A.

Suppose there are no free occurrences of x in A. Then $A_{x\to t} = A$ and, as α agrees with β on $\mathbf{free}(A)$, we have $\mathbf{t}_A(\alpha) = \mathbf{t}_A(\beta)$ so the Theorem holds in this case.

Suppose there are free occurrences of x in A. Then $w \neq x$ so

$$A_{x\to t} = \exists w\, B_{x\to t}.$$

We also have

$$\forall w\, \sim (B_{x\to t}) = \forall w\, (\sim B)_{x\to t} = (\forall w\, \sim B)_{x\to t}.$$

This implies

$$\mathbf{t}_{A_{x\to t}} = \mathbf{t}_{\exists w\, B_{x\to t}} = \sim \mathbf{t}_{\forall w \sim (B_{x\to t})} = \sim \mathbf{t}_{(\forall w \sim B)_{x\to t}}.$$

Since t is free for x in A we find that t is free for x in $A' = \forall w\, \sim B$. We need only apply the result of Part Two with A there equal A'.

Part Four. Suppose s_1, s_2 are terms and $A = (\, s_1 \,=\, s_2\,)$. Then

$$A_{x\to t} = (\,(s_1)_{x\to t} \,=\, (s_2)_{x\to t}\,)$$

so the Theorem holds in this case since

$$((s_i)_{x\to t})_\alpha = (s_i)_\beta, \quad i = 1, 2.$$

Part Five. Suppose (S5) holds. Then $A_{x\to t} = \sim (B_{x\to t})$ and t is free for x in B so by induction we have

$$\mathbf{t}_{A_{x\to t}}(\alpha) = \sim \mathbf{t}_{B_{x\to t}}(\alpha) = \sim \mathbf{t}_B(\beta) = \mathbf{t}_A(\beta).$$

Part Six. Suppose (S6) holds. Then t is free for x in B and C and $A_{x\to t} = (\, B_{x\to t}\, b\, C_{x\to t}\,)$ so, by induction,

$$\mathbf{t}_{A_{x\to t}}(\alpha) = \mathbf{t}_{B_{x\to t}}(\alpha)\, b\, \mathbf{t}_{C_{x\to t}}(\alpha) = \mathbf{t}_B(\beta)\, b\, \mathbf{t}_C(\beta) = \mathbf{t}_A(\beta).$$

$\square$