1. Gödel's Theorem.

1.1. ω-consistency.

Definition 1.1. We say formal number theory (FNT) is **ω-consistent** if whenever A is a statement, $x \in \mathbf{free}(A)$ and

$$\vdash A_{x \to \overline{k}} \quad \text{for all } k \in \mathbb{N}$$

it is *not* the case that

$$\vdash \exists x \sim A.$$

Mendelson remarks after this definition on page 142 of Mendelson that we accept that the standard interpretation as a model then FNT is ω-consistent.

Proposition 1.1. If FNT is ω-consistent then it is consistent.

Proof. Suppose FNT is ω consistent. Let $x \in X$ and let A be the statement $(x = x)$. Then, as we have seen,

$$\vdash A_{x \to \overline{k}} \quad \text{for all } k \in \mathbb{N}.$$

Since FNT is ω-consistent it is *not* the case that

$$\vdash \exists x \sim A.$$

Were FNT inconsistent we would have $\vdash B$ for all statements B. Thus FNT is consistent. □

1.2. The relation W_1.

Let g be the Gödel numbering function.

Let W_1 be the logical function of two arguments defined by requiring that $W_1(u, v) = 1$ if and only if there are a statement U and a finite sequence of statements V such that

(W1) $u = g(U)$ and $v = g(V)$;
(W2) $x_1 \in \mathbf{free}(U)$;
(W3) V is a proof of $U_{x_1 \to \overline{u}}$.

It will be shown that W_1 is primitive recursive; it follows immediately that W_1 is recursive. It will also be shown that any recursive function is expressible. Thus there is a statement A such that

$$\textbf{(E0)} \qquad \mathbf{free}(A) = \{x_1, x_2\};$$

$$\textbf{(E1)} \qquad W_1(k_1, k_2) = 1 \;\Rightarrow \vdash A_{x_1 \to \overline{k_1}, x_2 \to \overline{k_2}};$$

$$\textbf{(E2)} \qquad W_1(k_1, k_2) = 0 \;\Rightarrow \vdash \sim A_{x_1 \to \overline{k_1}, x_2 \to \overline{k_2}}.$$

Let B be the statement

$$\forall x_2 \sim A$$

and let C be the statement

$$B_{x_1 \to \overline{g(B)}}.$$

Note that C is

$$\forall x_2 \sim A_{x_1 \to \overline{g(B)}}.$$

We have

(1) $\qquad W(g(B), y) = 1 \;\Leftrightarrow\; y$ is the Gödel number of a proof of C.

Theorem 1.1. (Gödel's Theorem for FNT.) We have

(i) if FNT is consistent then it is not the case that $\vdash C$;

(ii) if FNT is ω-consistent the it is not the case that $\vdash\sim C$.

Proof. Suppose $\vdash C$. Then there is a proof P of C. We have $W_1(g(B), g(P)) = 1$ by (1). It follows that

$$\vdash A_{x_1 \to \overline{g(B)}, x_2 \to \overline{g(P)}}.$$

But by substitution of $\overline{g(P)}$ for x_2 in C we obtain

$$\vdash\sim A_{x_1 \to \overline{g(B)}, x_2 \to \overline{g(P)}}.$$

This implies FNT is inconsistent. Thus (i) holds.

Suppose FNT is ω-consistent and, contrary to (ii), $\vdash\sim C$. Since FNT is consistent by a preceding Proposition, it is not the case that $\vdash C$. By (1) we have

$$W_1(g(B), y) = 0 \quad \text{for all } y \in \mathbb{N}$$

and from (**E2**) this implies

$$\vdash\sim A_{x_1 \to \overline{g(B)}, x_2 \to \overline{y}} \quad \text{for all } y \in \mathbb{N}.$$

Since FNT is ω-consistent it is *not* the case that

$$\vdash \exists\, x_2 \;\sim\sim A_{x_1 \to g(B)}$$

which implies it is not the case that

$$\vdash \exists\, x_2\, A_{x_1 \to g(B)}$$

which is to say it is not the case that $\vdash\sim C$. This is a contradiction so (ii) holds.

$\square$