1. Scope.

Definition 1.1. Suppose A is an alphabet, $a \in A$, $s \in (A)^*$ and $n = |s|$. We let

$$\mathbf{o}(a, s) = \{i \in I(n) : s_i = a\}.$$

A member of $\mathbf{o}(a, s)$ is called an **occurrence of** a **in** s. We say a **occurs in** s if $\mathbf{o}(a, s) \neq \emptyset$.

Definition 1.2. Suppose S is a statement. B is **substatement of** S if B is a statement which is a substring of S.

Definition 1.3. Suppose S is a statement and $n = |S|$. We define

$$\mathbf{scope}_S : I(n) \to 2^{I(n)}$$

as follows. Let $\mathcal{Q} = (\mathcal{N}, \rho, p, <, f)$ be a parse tree for S and let

$$\lambda_0 < \lambda_1 < \cdots < \lambda_{n-1}$$

be the leaf nodes. Suppose $i \in I(n)$ and let ν be the parent of λ_i. Let $j \in I(n)$ and $m \in \mathbb{N}$ be such that λ_l, $j \leq l < j + m$ are the leaf nodes of the subtree with root node ν. Then $\mathbf{scope}_S(i) = \{l \in I(n) : j \leq l < j + m\}$.

Note that

$$i \in \mathbf{scope}_S(i) \quad \text{for } i \in I(n).$$

Proposition 1.1. Suppose S is a statement, $n = |S|$, $i \in I(n)$, $j, m \in \mathbb{N}$ are such that

$$\mathbf{scope}_S(i) = \{l \in \mathbb{N} : j \leq l < j + m\}$$

and

$$T = \mathbf{s}_{j,m}(S).$$

Then exactly one of the following holds:

(i) for some $u \in X \cup C$,

$$T = u$$

(ii) for some $u, v \in X \cup C$ and some $o \in O$,

$$T = (\ u\ o\ v\ )$$

(iii) for some $n \in \mathbb{N}^+$, some $s \in F_n \cup R_n$ and some $u_1, \ldots, u_n \in X \cup C$,

$$T = s(u_1, u_2, \ldots, u_n)$$

(iv) for some $q \in \{\forall, \exists\}$, some $x \in X$ and some statement U,

$$T = q\ x\ U$$

Proof. Study a parse tree for S. □

Definition 1.4. Suppose S is a statement and $n = |S|$. We define

$$\mathbf{q}_S : I(n) \to 2^{I(n)}$$

at $i \in I(n)$ by letting $\mathbf{q}_S(i)$ be the set of those $j \in I(n)$ such that

$$S_j \in \{\forall, \exists\} \quad \text{and} \quad i \in \mathbf{scope}_S(j).$$

Definition 1.5. Suppose S is a statement and $x \in X$. We say an occurrence i of x in S is **bound** if

$$S_{j+1} = x \quad \text{for some } j \in \mathbf{q}_S(i).$$

We say an occurrence i of x is **free** if it is not bound.

We let

$$\mathbf{free}(S) = \{x \in X : \text{ there is a free occurrence of } x \text{ in } S\}$$

and we let

$$\mathbf{bound}(S) = \{x \in X : \ x \text{ occurs in } S \text{ and } x \notin \mathbf{free}(S)\}.$$

Definition 1.6. Suppose S is a statement, t is a term and x is a variable. We say t **is free for** x **in** S if, for each $j \in \mathbf{q}_S$ such that S_{j+1} occurs in t, no free occurrence of x belongs to $\mathbf{scope}_S(j)$.

(See Mendelson p. 48 or Hodel p. 161 where the term "substitutable" is used.)