1. Back to formal theories.

I'm going to modify some notions involving formal theories. Suppose

$$\mathcal{T} = (L, A, \mathcal{R})$$

is a formal theory where L is a language on the alphabet A and $\mathcal{R}$ is the set of rules of inference.

Definition 1.1. Suppose $\Gamma \subset L$.

We say the finite sequence

$$A_1, A_2 \ldots, A_n$$

is a **primary proof using** Γ if for each $j \in \{1, \ldots, n\}$ *either* $A_j \in \Gamma$ *or* there is a rule of inference $(\mathcal{H}, C)$ such that $\mathcal{H} \subset \{A_i : i < j\}$ and $C = A_j$. We say the statement A is a **theorem (of $\mathcal{T}$) using** Γ if there is a primary proof $A_1, \ldots, A_n$ using Γ such that $A_n = A$ in which case we write

$$\Gamma \vdash A.$$

We say the finite sequence

$$A_1, A_2 \ldots, A_n$$

is a **proof using** Γ if for each $j \in \{1, \ldots, n\}$ *either* $A_j \in \Gamma$, *or* $\Gamma \cup \{A_i : i < j\} \vdash A_j$ *or* there is a rule of inference $(\mathcal{H}, C)$ such that $\mathcal{H} \subset \{A_i : i < j\}$ and $C = A_j$.

Proposition 1.1. Suppose $\Gamma \subset L$ and $A \in L$ then $\Gamma \vdash A$ if and only if there is a proof $A_1, A_2, \ldots, A_n$ using Γ such that $A_n = A$.

Proof. Just replace each A_j such that $\Gamma \cup \{A_i : i < j\} \vdash A_j$ be a primary proof using $\Gamma \cup \{A_i : i < j\}$ whose last statement is A_j. □

2. A very useful technique.

Let us fix a first order logic $\mathcal{F}$. Let $\mathcal{S}$ be the set of statements.

Definition 2.1. Suppose P is a set of propositional variables,

$$\sigma : P \to \mathcal{S}$$

and $A \in \mathbf{l}(P)$. Let

$$A_\sigma$$

be the the string obtained by replacing each occurrence of $p \in P$ in A by $\sigma(p)$.

Proposition 2.1. Suppose P is a set of propositional variables,

$$\sigma : P \to \mathcal{S}$$

and $A \in \mathbf{l}(P)$.

Then $A_\sigma \in \mathcal{S}$. Moreover, for any $A, B \in \mathbf{p}(P)$ we have

$$(\sim A)_\sigma = \sim (A_\sigma)$$

and

$$(A \, o \, B)_\sigma = (A_\sigma \, o \, B_\sigma) \quad \text{whenever } o \in \{\vee, \wedge, \to, \leftrightarrow\}.$$

Proof. Extend a parse tree for A by replacing the nodes corresponding to $p \in P$ with a parse tree for $\sigma(p)$. □

Proposition 2.2. Suppose P, σ and Σ are as in the preceding Proposition, $\Gamma \subset \mathbf{p}(P)$, $B \in \mathbf{p}(P)$ and

$$\Gamma \vdash B.$$

Then

$$\{A_\sigma : A \in \Gamma\} \vdash B_\sigma.$$

Proof. One only has to check that if $(\mathcal{H}, C)$ is a rule of inference for propositional logic then

$$\{H_\sigma : H \in \mathcal{H}\} \vdash C_\sigma;$$

this follows directly from the preceding Proposition. □

Theorem 2.1. Suppose P is a set of propositional variables,

$$\sigma, \tau : P \to \mathcal{S}$$

and $A \in \mathbf{l}(P)$. Then

$$\{(\sigma(p) \leftrightarrow \tau(p)) : p \in P\} \vdash A_\sigma \leftrightarrow B_\tau.$$

Proof. We have the following Lemma.

Lemma 2.1. Suppose $B, C, D, E \in \mathcal{S}$. Then

$$\{(B_\sigma \leftrightarrow C_\sigma)\} \vdash (\sim B)_\sigma \leftrightarrow (\sim C)_\sigma$$

and

$$\{(B_\sigma \leftrightarrow C_\sigma), (D_\sigma \leftrightarrow E_\sigma)\} \vdash ((B_\sigma \, o \, D_\sigma) \leftrightarrow (D_\sigma \, o \, E_\sigma))$$

whenever $o \in \{\vee, \wedge, \to, \leftrightarrow\}$.

Proof. Combine the preceding Proposition with the following theorems from propositional logic:

$$\{(p \leftrightarrow q)\} \vdash (\sim p \leftrightarrow \sim q), \quad \{(p \leftrightarrow q), (r \leftrightarrow s)\} \vdash ((p \, o \, r) \leftrightarrow (q \, o \, s))$$

where $o \in \{\vee, \wedge, \to, \leftrightarrow\}$. □

To prove the Theorem we use induction on the the depth of a parse tree for A making use of the Lemma to carry out the inductive step. □