1. Back to formal theories.

I’m going to modify some notions involving formal theories. Suppose

\[ T = (L, A, R) \]

is a formal theory where \( L \) is a language on the alphabet \( A \) and \( R \) is the set of rules of inference.

**Definition 1.1.** Suppose \( \Gamma \subset L \).

We say the finite sequence

\[ A_1, A_2, \ldots, A_n \]

is a **primary proof using** \( \Gamma \) if for each \( j \in \{1, \ldots, n\} \) either \( A_j \in \Gamma \) or there is a rule of inference \((H, C)\) such that \( H \subset \{A_i : i < j\} \) and \( C = A_j \). We say the statement \( A \) is a **theorem (of \( T \)) using** \( \Gamma \) if there is a primary proof \( A_1, \ldots, A_n \) using \( \Gamma \) such that \( A_n = A \) in which case we write

\[ \Gamma \vdash A. \]

We say the finite sequence

\[ A_1, A_2, \ldots, A_n \]

is a **proof using** \( \Gamma \) if for each \( j \in \{1, \ldots, n\} \) either \( A_j \in \Gamma \), or \( \Gamma \cup \{A_i : i < j\} \vdash A_j \) or there is a rule of inference \((H, C)\) such that \( H \subset \{A_i : i < j\} \) and \( C = A_j \).

**Proposition 1.1.** Suppose \( \Gamma \subset L \) and \( A \in L \) then \( \Gamma \vdash A \) if and only if there is a proof \( A_1, A_2, \ldots, A_n \) using \( \Gamma \) such that \( A_n = A \).

**Proof.** Just replace each \( A_j \) such that \( \Gamma \cup \{A_i : i < j\} \vdash A_j \) be a primary proof using \( \Gamma \cup \{A_i : i < j\} \) whose last statement is \( A_j \). \( \square \)

2. A very useful technique.

Let us fix a first order logic \( F \). Let \( S \) be the set of statements.

**Definition 2.1.** Suppose \( P \) is a set of propositional variables,

\[ \sigma : P \to S \]

and \( A \in \mathbb{L}(P) \). Let

\[ A_\sigma \]

be the the string obtained by replacing each occurrence of \( p \in P \) in \( A \) by \( \sigma(p) \).

**Proposition 2.1.** Suppose \( P \) is a set of propositional variables,

\[ \sigma : P \to S \]

and \( A \in \mathbb{L}(P) \).

Then \( A_\sigma \in S \). Moreover, for any \( A, B \in \mathbb{P}(P) \) we have

\[ (~ A)_\sigma = ~ (A_\sigma) \]

and

\[ (A \circ B)_\sigma = (A_\sigma \circ B_\sigma) \quad \text{whenever } o \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}. \]

**Proof.** Extend a parse tree for \( A \) by replacing the nodes corresponding to \( p \in P \) with a parse tree for \( \sigma(p) \). \( \square \)
Proposition 2.2. Suppose $P$, $\sigma$ and $\Sigma$ are as in the preceding Proposition, $\Gamma \subseteq p(P)$, $B \in p(P)$ and
$$\Gamma \vdash B.$$ 
Then
$$\{A_{\sigma} : A \in \Gamma\} \vdash B_{\sigma}.$$ 

Proof. One only has to check that if $(H, C)$ is a rule of inference for propositional logic then
$$\{H_{\sigma} : H \in \mathcal{H}\} \vdash C_{\sigma};$$ 
this follows directly from the preceding Proposition. □

Theorem 2.1. Suppose $P$ is a set of propositional variables,
$$\sigma, \tau : P \rightarrow S$$
and $A \in l(P)$. Then
$$\{(\sigma(p) \leftrightarrow \tau(p)) : p \in P\} \vdash A_{\sigma} \leftrightarrow B_{\tau}.$$

Proof. We have the following Lemma.

Lemma 2.1. Suppose $B, C, D, E \in S$. Then
$$\{(B_{\sigma} \leftrightarrow C_{\sigma})\} \vdash (\sim B)_{\sigma} \leftrightarrow (\sim C)_{\sigma}$$
and
$$\{(B_{\sigma} \leftrightarrow C_{\sigma}), (D_{\sigma} \leftrightarrow E_{\sigma})\} \vdash (B_{\sigma} \circ D_{\sigma}) \leftrightarrow (D_{\sigma} \circ E_{\sigma})$$
whenever $o \in \{\lor, \land, \rightarrow, \leftrightarrow\}$.

Proof. Combine the preceding Proposition with the following theorems from propositional logic:
$$\{(p \leftrightarrow q)\} \vdash (\sim p \leftrightarrow \sim q), \quad \{(p \leftrightarrow q), (r \leftrightarrow s)\} \vdash ((p \circ r) \leftrightarrow (q \circ s))$$
where $o \in \{\lor, \land, \rightarrow, \leftrightarrow\}$. □

To prove the Theorem we use induction on the the depth of a parse tree for $A$ making use of the Lemma to carry out the inductive step. □