1. Expressibility and representability.

Definition 1.1. (Unique existence.) Suppose x is a variable and A is a statement in a first order language. We write

$$\exists!\, x\, A$$

to mean

$$(\exists\, x\, A \wedge \forall\, x \,\forall\, y\, ((A \wedge A_{x \to y}) \to (x = y)))$$

for all variables y which do not occur in A.

Definition 1.2. (See page 117 in Mendelson.) Suppose $R \in \mathbb{N}^n_1$. We say R is **logical** if $R(x) \in \{0, 1\}$ whenever $x \in \mathbb{N}^n$. If R is logical we say R is **expressible** if there is a statement A such that

$$(\mathbf{E0}) \qquad \mathbf{free}(A) = \{x_1, \ldots, x_n\}$$

and, for any $k \in \mathbb{N}^n$, we have

$$(\mathbf{E1}) \qquad R(k) = 1 \;\Rightarrow \vdash A_{x \to \overline{k}}$$

and

$$(\mathbf{E2}) \qquad R(k) = 0 \;\Rightarrow \vdash\sim A_{x \to \overline{k}}$$

where we have set $x = (x_1, \ldots, x_n)$ and $\overline{k} = (\overline{k_1}, \ldots, \overline{k_{n+1}})$.

Definition 1.3. (See page 118 in Mendelson.) Suppose $F \in \mathbb{N}^n_1$. We say F is **representable** if there is a statement B such that

$$(\mathbf{R0}) \qquad \mathbf{free}(B) = \{x_1, \ldots, x_n, x_{n+1}\}$$

and, for any $k \in \mathbb{N}^n$ and $l \in \mathbb{N}$, we have

$$(\mathbf{R1}) \qquad F(k) = l \;\Rightarrow \vdash B_{x \to \overline{k}, x_{n+1} \to \overline{l}l}$$

and

$$(\mathbf{R2}) \qquad \vdash \exists!\, x_{n+1} B_{x \to k}$$

where we have set $x = (x_1, \ldots, x_n)$ and $\overline{k} = (\overline{k_1}, \ldots, \overline{k_{n+1}})$.

We say F is **strongly representable** if (**R1**) holds and we have

$$(\mathbf{SR2}) \qquad \vdash \exists!\, x_{n+1} B$$

Proposition 1.1. Suppose $m, n \in \mathbb{N}$. Then

(i) $\vdash (\overline{m} + \overline{n} = \overline{m + n})$;
(ii) $\vdash (\overline{m} \cdot \overline{n} = \overline{mn})$;
(iii) $m \neq n \;\Rightarrow \sim (\overline{m} = \overline{n})$;

Proof. See page 110 in Mendelson. It's not that hard; one needs to set up inductions. □

Proposition 1.2. Suppose $R \in \mathbb{N}^n$ and R is logical. Then

(i) if R is expressible then R is strongly representable
(ii) if R is representable then R is expressible.

Proof. Let $x = (x_1, \ldots, x_n)$.

Proof of (i). Suppose R is is expressible and A is as in (**E0-E2**). Let B be the statement

$$((A \wedge (x_{n+1} = \overline{1})) \vee (\sim A \wedge (x_{n+1} = \overline{0})))$$

Then (**R0**) holds. Suppose $k \in \mathbb{N}^n$, $l \in \mathbb{N}$ and $R(k) = l$. Then $l \in \{0, 1\}$ since R is logical. Keeping in mind that

$$\vdash \sim (\bar{l} = \overline{0}) \quad \text{if } l \neq 0$$

we find that

$$\begin{aligned} B_{x \to \overline{k}, x_{n+1} \to \bar{l}} &= ((A_{x \to \overline{k}} \wedge (\bar{l} = \overline{1})) \vee (\sim A_{x \to \overline{k}} \wedge (\bar{l} = \overline{0}))) \\ &\leftrightarrow \begin{cases} A_{x \to \overline{k}} & \text{if } l = 1, \\ \sim A_{x \to \overline{k}} & \text{if } l = 0 \end{cases} \end{aligned}$$

Thus

$$\vdash B_{x \to \overline{k}, x_{n+1} \to \bar{l}}$$

Thus (**R1**) holds as does $\vdash \forall x_{n+1} B$. Now

$$B \wedge B_{x_{n+1} \to x_{n+2}}$$

is the disjunction of

$$\begin{aligned} &A \wedge (x_{n+1} = \overline{1}) \wedge A \wedge (x_{n+2} = \overline{1}) \\ &A \wedge (x_{n+1} = \overline{1}) \wedge \sim A \wedge (x_{n+2} = \overline{0}) \\ &\sim A \wedge (x_{n+1} = \overline{0}) \wedge A \wedge (x_{n+2} = \overline{1}) \\ &\sim A \wedge (x_{n+1} = \overline{0}) \wedge \sim A \wedge (x_{n+2} = \overline{0}) \end{aligned}$$

which is equivalent to

$$(A \wedge (x_{n+1} = \overline{1}) \wedge (x_{n+2} = \overline{1})) \vee (\sim A \wedge (x_{n+1} = \overline{0}) \wedge (x_{n+2} = \overline{0}))$$

which by the cut rule implies

$$((x_{n+1} = \overline{1}) \wedge (x_{n+2} = \overline{1})) \vee ((x_{n+1} = \overline{0}) \wedge (x_{n+2} = \overline{0}))$$

which implies

$$(x_{n+1} = x_{n+2})$$

That is,

$$\{B \wedge B_{x_{n+1} \to x_{n+2}}\} \vdash (x_{n+1} = x_{n+2})$$

and the proof did not use the **Add** $\forall$ rule. By the Deduction Theorem we obtain

$$\vdash (B \wedge B_{x_{n+1} \to x_{n+2}}) \to (x_{n+1} = x_{n+2})$$

Finally, two applications of **Gen** allows us to conclude that (**SR2**) holds.

Proof of (ii). Suppose R is representable and B is as in (**R0-R2**). Let A be the statement

$$B_{x_{n+1} \to 1}$$

Then (**E0**) holds. Let $k \in \mathbb{N}^n$

Suppose $R(k) = 1$. Then, by (**R1**),

$$\vdash B_{x \to \overline{k}, x_{n+1} \to \overline{1}}$$

Since $B_{x \to \overline{k}, x_{n+1} \to \overline{1}} = A_{x \to \overline{k}}$ we find that

$$\vdash A_{x \to k}$$

Suppose $R(k) = 0$. Then, by (**R1**),

$$\vdash B_{x\to\overline{k},x_{n+1}\to\overline{0}}$$

Since (**R2**) holds we infer from the substitution axiom (applied to x_{n+1} and x_{n+2} that

$$((B_{x\to k})_{x_{n+1}\to\overline{0}} \wedge ((B_{x\to k})_{x_{n+1}\to x_{n+2}})_{x_{n+2}\to\overline{1}}) \to (\overline{1} = \overline{0}).$$

Since

$$(B_{x\to k})_{x_{n+1}\to\overline{0}} \quad \text{is} \quad B_{x\to k,x_{n+1}\to\overline{0}},$$

since

$$((B_{x\to k})_{x_{n+1}\to x_{n+2}})_{x_{n+2}\to\overline{1}} \quad \text{is} \quad A_{x\to\overline{k}}$$

and since

$$\vdash\sim (\overline{1} = \overline{0})$$

we find that

$$\vdash\sim (B_{x\to k,x_{n+1}\to\overline{0}} \wedge A_{x\to\overline{k}}).$$

But

$$\vdash B_{x\to k,x_{n+1}\to\overline{0}}$$

by (**R1**) so

$$\vdash\sim A_{x\to\overline{k}}$$

as desired. □