1. Some examples from propositional logic.

$\{A, (\sim A \vee B)\} \vdash B$

A
$(A \vee B)$
$(\sim A \vee B)$
$(B \vee B)$
B

$\{\sim A, (A \vee B)\} \vdash B$

$\sim A$
$(\sim A \vee B)$
$(A \vee B$
$(B \vee B)$
B

$\{A \wedge B\} \vdash \sim (\sim A \vee \sim B)$

$(A \wedge B)$
$\sim (A \wedge B) \vee \sim (\sim A \vee \sim B)$
$\sim (\sim A \vee \sim B)$

$\{(A \wedge B)\} \vdash B$

$(A \wedge B)$
$\sim (\sim A \vee \sim B)$
$(\sim B \vee B)$
$(\sim A \vee (\sim B \vee B))$
$((\sim A \vee \sim B) \vee B)$
B

$$\{(A \wedge B)\} \vdash A$$

$$(A \wedge B)$$
$$\sim (\sim A \vee \sim B)$$
$$(A \vee \sim A)$$
$$((A \vee \sim A) \vee \sim B)$$
$$(A \vee (\sim A \vee \sim B))$$
$$((\sim A \vee \sim B) \vee A)$$
$$A$$

2. An example of logical validity and invalidity.

2.1. **Exactly one variable.** Suppose $X = \{x\}$; A and B are statements; and $\mathcal{I} = (D, \mathbf{C}, \mathbf{F}, \mathbf{R})$ is an interpretation. (It is trivial that an interpretation exists). Then

$$\mathbf{t}_{\forall\, x\, A} = \mathbf{t}_A$$

because if $\beta \in D^X$ then $\beta \sim_x \alpha$ iff $\beta = \alpha$. Thus

$$\forall\, x\, (A \vee B) \leftrightarrow (\forall\, x\, A \vee \forall\, x\, B)$$

is logically valid.

2.2. **At least two variables.** Suppose $x \in X$; A is a statement and $B =\sim A$. Then

$$\mathbf{t}_{(A \vee B)} = \mathbf{t}_A \vee \sim \mathbf{t}_A = 1$$

so $\forall\, x\, (A \vee B)$ is logically valid.

Suppose $y \in X$, $y \neq x$; and $\mathcal{I} = (D, \mathbf{C}, \mathbf{F}, \mathbf{R})$ is an interpretation such that D has a least two members. (It is trivial that such an interpretation exists.)

Let A be the statement $(x = y)$; thus B is the statement $\sim (x = y)$. Let $\alpha \in D^X$ be such that $\alpha(x) \neq \alpha(y)$; such and α exists because $x \neq y$ and D has at least two members.

Then $\alpha \sim_x \alpha$ and

$$\mathbf{t}_A(\alpha) = \mathbf{t}_{(x=y)}(\alpha) = 0$$

so $\mathbf{t}_{\forall\, x\, A}(\alpha) = 0$.

Let $\beta \in D^X$ be such that $\beta(x) = \alpha(x)$ and $\beta(y) = \alpha(x)$. Then $\beta \sim_x \alpha$ and

$$\mathbf{t}_B(\beta) = \mathbf{t}_{\sim(x=y)}(\beta) = 0$$

so $\mathbf{t}_{\forall\, x\, B}(\alpha) = 0$.

It follows that $\mathbf{t}_{\forall\, x\, A \vee \forall\, x\, B}(\alpha) = 0$ so that

$$\forall\, x\, (A \vee B) \leftrightarrow (\forall\, x\, A \vee \forall\, x\, B)$$

is not true in $\mathcal{I}$.