Independent families of random variables.

**Definition.** Suppose $\mathcal{X}$ is a family of random variables. We say $\mathcal{X}$ is independent if

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n)$$

whenever $X_1, \ldots, X_n$ are distinct members of $\mathcal{X}$ and $A_1, \ldots, A_n$ are Borel subsets of $\mathbb{R}$.

**Proposition.** Suppose $\mathcal{X}$ is a family of random variables. Then $\mathcal{X}$ is independent if and only if

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

depending whenever $X_1, \ldots, X_n$ are distinct members of $\mathcal{X}$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

**Proof.** This is really simple once you get straight what a Borel set is. We won’t do this, though. □

**Proposition.** Suppose $X_1, \ldots, X_n$ are distinct discrete random variables. Then $\{X_1, \ldots, X_n\}$ is independent if and only if

(1) $p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$ whenever $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

**Proof.** This is long winded but simple minded. I hope you will see this.

Suppose $\{X_1, \ldots, X_n\}$ is independent and $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $A_i = \{x_i\}$ for each $i = 1, \ldots, n$. Then

$$p_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$$

$$= P(X_1 \in A_1, \ldots, X_n \in A_n)$$

$$= P(X_1 \in A_1) \cdots P(X_n \in A_n)$$

$$= P(X_1 = x_1) \cdots P(X_n = x_n)$$

$$= p_{X_1}(x_1) \cdots p_{X_n}(x_n);$$

thus (1) holds.

One the other hand suppose (1) holds and $A_i \subset \mathbb{R}$, $i = 1, \ldots, n$. Then

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(\bigcup_{x_1 \in A_1, \ldots, x_n \in A_n} \{X_1 = x_1, \ldots, X_n = x_n\})$$

$$= \sum_{x_1 \in A_1, \ldots, x_n \in A_n} P(X_1 = x_1, \ldots, X_n = x_n)$$

$$= \sum_{x_1 \in A_1, \ldots, x_n \in A_n} p_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

$$= \sum_{x_1 \in A_1, \ldots, x_n \in A_n} p_{X_1}(x_1) \cdots p_{X_n}(x_n)$$

$$= \left(\sum_{x_1 \in A_1} p_{X_1}(x_1)\right) \cdots \left(\sum_{x_n \in A_n} p_{X_n}(x_n)\right)$$

$$= \left(\sum_{x_1 \in A_1} P(X_1 = x_1)\right) \cdots \left(\sum_{x_n \in A_n} P(X_n = x_n)\right)$$

$$= (P(\bigcup_{x_1 \in A_1} \{X_1 = x_1\}) \cdots (P(\bigcup_{x_n \in A_n} \{X_n = x_n\}))$$

$$= P(X_1 \in A_1) \cdots P(X_n \in A_n)$$
so \(\{X_1, \ldots, X_n\}\) is independent. □

**Definition.** We say the random vector \(X = (X_1, \ldots, X_n)\) is **continuous** if there exists a function

\[
f_X = f_{X_1, \ldots, X_n} : \mathbb{R}^n \to [0, \infty),
\]

called a **(joint) probability density function**, such that

\[
F_X(x) = F_{X_1, \ldots, X_n}(x_1, \ldots, x_n)
\]

\[
= \int_{w \leq x} f_X(w) \, dw
\]

\[
= \int \cdots \int_{w_1 \leq x_1, \ldots, w_n \leq x_n} f_{X_1, \ldots, X_n}(w_1, \ldots, w_n) \, dw_1 \cdots dw_n
\]

\[
= \int_{-\infty}^{x_1} \cdots \left( \int_{-\infty}^{x_n} f_{X_1, \ldots, X_n}(w_1, \ldots, w_n) \, dw_1 \right) \cdots dw_n.
\]

We use the following formula **very** frequently.

**Theorem.** Suppose \(X = (X_1, \ldots, X_n)\) is continuous random vector and \(R\) is a Borel subset of \(\mathbb{R}^n\). Then

\[
P(X \in R) = P((X_1, \ldots, X_n) \in R) = \int_R f_X(x) \, dx = \int \cdots \int_R f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]

**Proof.** This is straight forward but technical exercise which we omit. □

**Corollary.** Suppose \(X = (X_1, \ldots, X_n)\) is continuous random vector and \(1 \leq i_1 < \cdots < i_m \leq n\). Then \((X_{i_1}, \ldots, X_{i_m})\) is a continuous random vector for which

\[
f_{X_{i_1}, \ldots, X_{i_m}}(x_{i_1}, \ldots, x_{i_m})
\]

equals the integral over all of the other variables.

**Definition.** Suppose \(R\) is a Borel set in \(\mathbb{R}^n\). We say the random vector \(X = (X_1, \ldots, X_n)\) is **uniformly distributed over** \(R\) if

\[
P(X \in Q) = \frac{|Q \cap R|}{|R|}
\]

whenever \(Q\) is a Borel subset of \(\mathbb{R}^n\).

It is a straightforward but technical exercise which we omit to show that \(X\) is continuous with pdf given by

\[
f_X(x) = \begin{cases} \frac{1}{|R|} & \text{if } x \in R, \\ 0 & \text{else.} \end{cases}
\]