Transformations of continuous random vectors.

**Theorem.** Suppose

1. $U$ is an open subset of $\mathbb{R}^n$ and $X = (X_1, \ldots, X_n)$ is a continuous random vector with range $U$;
2. $V$ is an open subset of $\mathbb{R}^n$ and $g = (g_1, \ldots, g_n) : U \to V$ is a continuously differentiable mapping carrying $U$ one-to-one onto $V$ with continuously differentiable inverse and
3. $Y = g(X)$.

Then $Y$ is continuous and

$$f_Y(y) = \begin{cases} f_X(x) \left| \det \frac{\partial g}{\partial x}(x) \right|^{-1} & \text{if } x \in U \text{ and } y = g(x), \\ 0 & \text{else.} \end{cases}$$

**Proof.** Suppose $x \in U$ and $y = g(x)$. For each positive $h$ let

$$C_h = (y_1 - h, y_1 + h) \times \cdots \times (y_n - h, y_n + h).$$

Then

$$f_Y(y) = \lim_{h \downarrow 0} \frac{P(Y \in C_h)}{(2h)^n}$$

$$= \lim_{h \downarrow 0} \frac{P(X \in g^{-1}(C_h))}{(2h)^n}$$

$$= \lim_{h \downarrow 0} \frac{1}{(2h)^n} \int_{g^{-1}(C_h)} f_X(w) \, dw$$

$$= f_X(x) \lim_{h \downarrow 0} \frac{1}{(2h)^n} \int_{g^{-1}(C_h)} dw$$

$$= f_X(x) \left| \det \frac{\partial g}{\partial x}(x) \right|^{-1}.$$