The average on this test was 102.07 (out of 140 possible points). The standard deviation was 25.29.

1. 5 points. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is such that

\[
f(x) = \begin{cases} 
1 & \text{if } 0 < x < 1/2, \\
1 & \text{if } 3/2 < x < 5/2, \\
0 & \text{else}.
\end{cases}
\]

Indicate why \( f \) cannot be the probability density function of a continuous random variable.

Solution. We have

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = \int_{0}^{\frac{1}{2}} dx + \int_{\frac{3}{2}}^{\frac{5}{2}} dx = \frac{1}{2} + 1 \neq 1.
\]

2. 10 points. Let \((X, Y)\) be a discrete random vector such that

\[
p_{X,Y}(x, y) = \begin{cases} 
\frac{1}{6} & \text{if } x = 1, 2, 3 \text{ and } y = 1, 2, \\
0 & \text{else}.
\end{cases}
\]

(a) Indicate whether \(X\) and \(Y\) are independent and indicate how you arrived at your answer. (b) Compute \(E(XY)\).

Solution. (a) We have

\[
p_X(x) = \sum_y p_{X,Y}(x, y) = \begin{cases} 
\frac{1}{6} + \frac{1}{6} = \frac{1}{3} & \text{if } x = 1, 2, 3, \\
0 & \text{else};
\end{cases}
\]

\[
p_Y(y) = \sum_x p_{X,Y}(x, y) = \begin{cases} 
\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} & \text{if } y = 1, 2, \\
0 & \text{else};
\end{cases}
\]

\[
p_X(x)p_Y(y) = \begin{cases} 
\frac{1}{3} & \text{if } x = 1, 2, 3, y = 1, 2, \\
0 & \text{else}.
\end{cases}
\]

Thus

\[
p_{X,Y}(x, y) = p_X(x)p_Y(y) \quad \text{for all } x, y
\]

so \(X\) and \(Y\) are independent.

(b) We have

\[
E(X) = \sum_x x p_X(x) = \frac{1}{3} (1 + 2 + 3) = 2 \quad \text{and} \quad E(Y) = \sum_y y p_Y(y) = \frac{1}{2} (1 + 2) = \frac{3}{2}
\]

so, as \(X\) and \(Y\) are independent, we have

\[
E(XY) = E(X)E(Y) = 3.
\]

3. 5 points. Suppose \(X\) is uniform on \((0, 2)\). Compute \(P(1/2 < X < 1) \cup \{3/2 < X < 5/2\})\).
Solution. Since $P(X \geq 2) = 0$ we have

\[
P(\{1/2 < X < 1\} \cup \{3/2 < X < 5/2\}) = P(1/2 < X < 1) + P(3/2 < X < 5/2) \\
= P(1/2 < X < 1) + P(3/2 < X < 2) \\
= \frac{1}{2} - \frac{1}{2} + \frac{2 - \frac{3}{2}}{2 - 0} \\
= \frac{1}{2}.
\]

4. 10 points. Compute the expectation and variance of $X$ where $X$ is a continuous random variable such that

\[
f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x < 1, \\ 0 & \text{else}. \end{cases}
\]

Solution. We have

\[
E(X) = \int_{-\infty}^{\infty} xf_X(x) \, dx = \int_{0}^{1} x \frac{dx}{2\sqrt{x}} = \frac{1}{2} \left[ \frac{1}{2} x^{\frac{3}{2}} \right]_{0}^{1} = \frac{1}{3}; \\
E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{0}^{1} x^2 \frac{dx}{2\sqrt{x}} = \frac{1}{2} \left[ \frac{1}{2} x^{\frac{5}{2}} \right]_{0}^{1} = \frac{1}{5}; \\
Var(X) = E(X^2) - E(X)^2 = \frac{1}{5} - \left( \frac{1}{3} \right)^2 = \frac{4}{45}.
\]

5. 10 points. Use the normal approximation to the binomial to approximately calculate the probability that in 1600 flips of a fair coin at most 830 are heads.

Solution. Let $X$ be binomial with $n = 1600$ and $p = 1/2$. Then $\mu = E(X) = np = 800$ and $\sigma = \sqrt{Var(X)} = \sqrt{np(1-p)} = \sqrt{1600 \cdot (1/2)(1-(1/2))} = 20$. Let $Z$ be standard normal. Then

\[
P(X \leq 830) = P(X < 830.5) \\
= P \left( \frac{X - \mu}{\sigma} < \frac{830.5 - \mu}{\sigma} \right) \\
\approx P(Z < 1.525) \\
\approx .9364.
\]

6. 10 points. Twelve percent of the population is lefthanded. Approximately compute the probability that there are at least 20 lefthanders in a school of 200 students. State your assumptions.

Solution. We assume that the draws from the population are independent and that the probability of picking a lefthander is .12. So let $X$ be binomial with $n = 200$ and $p = .12$. We have $\mu = E(X) = 24$ and $\sigma = \sqrt{Var(X)} = \sqrt{np(1-p)} = \sqrt{200 \cdot .12 \cdot .88} \approx .460$.

Let $Z$ be standard normal. Then

\[
P(X \geq 20) = P(X > 19.5) \\
= P \left( \frac{X - \mu}{\sigma} > \frac{19.5 - \mu}{\sigma} \right) \\
\approx P(Z > .9791) \\
= P(Z < .9791) \\
\approx .8365.
\]
7. 10 points. Suppose the random variable $X$ is exponentially distributed with parameter $\lambda > 0$. (a) Compute the cdf of $X^2$. (b) Decide whether or not $X^2$ is continuous and give a reason for your answer.

Solution. Let $Z = X^2$. Then

\[
\text{rng } Z = \{ x^2 : x \in \text{rng } X \} = \{ x^2 : x \in (0, \infty) \} = (0, \infty)
\]
so $F_Z(z) = 0$ if $z \leq 0$. Suppose $z > 0$. Then

\[
F_Z(z) = P(X^2 \leq z) = P(X \leq \sqrt{z}) \quad (\text{because } X \text{ and } z \text{ are positive}) \quad 1 - e^{-\lambda \sqrt{z}}.
\]

(b) $Z$ is continuous because if we let

\[
f(z) = F'_Z(z) = \begin{cases} \frac{\lambda e^{-\sqrt{z}}}{2 \sqrt{z}} & \text{if } z > 0, \\ 0 & \text{else} \end{cases}
\]
we find that

\[
F_Z(z) = \int_{-\infty}^{z} f(w) \, dw.
\]

8. 20 points. Suppose $(X, Y)$ is uniformly distributed on the triangle

\[
\{(x, y) \in \mathbb{R}^2 : 0 < x, \ 0 < y, \ x + y < 1\}.
\]

(a) Compute the cdf of $X + Y$. (b) Compute $E(X)$. (c) Compute $E(XY)$.

Solution. (a) Let $T = \{(x, y) \in \mathbb{R}^2 : 0 < x, \ 0 < y \text{ and } x + y < 1\}$; since the area of $T$ is $\frac{1}{2}$ we have

\[
f_{X,Y}(x, y) = \begin{cases} 2 & \text{if } (x, y) \in T, \\ 0 & \text{else}. \end{cases}
\]

Let $Z = X + Y$. Then $\text{rng } Z = \{ x+y : (x, y) \in T \} = (0, 1)$ so $F_Z(z) = 0$ if $z \leq 0$ and $F_Z(z) = 1$ if $z \geq 1$. If $0 < z < 1$ we have

\[
F_Z(z) = P(X + Y \leq z)
\]

\[
= \int \int_{x+y \leq z} f_{X,Y}(x, y) \, dxdy
\]

\[
= 2 \text{Area}(\{(x, y) \in T : x + y \leq z\})
\]

\[
= z^2.
\]

(b) $E(X) = \int \int_{\mathbb{R}^2} xf_{X,Y}(x, y) \, dxdy = 2 \int_0^1 \left( \int_0^{1-x} x \, dy \right) \, dx = \frac{1}{3}$.

(c) $E(XY) = \int \int_{\mathbb{R}^2} xyf_{X,Y}(x, y) \, dxdy = 2 \int_0^1 \left( \int_0^{1-x} xy \, dy \right) \, dx = \frac{1}{12}$.

9. 15 points. Suppose $X$ is uniformly distributed on $(-1, 2)$. Calculate the cdf of $X^2$. 


Solution. This is tricky. Let $Z = X^2$. We have

$$\text{rng } Z = \{x^2 : x \in \text{rng } X\} = \{x^2 : x \in (-1, 2)\} = [0, 4).$$

Thus $F_Z(z) = 0$ if $z \leq 0$ and $F_Z(z) = 1$ if $x \geq 4$. Suppose $0 < z < 4$. Then

$$P(-X \leq \sqrt{z}, X < 0) = \frac{1}{3} \begin{cases} 1 & \text{if } z \geq 1, \\ \sqrt{z} & \text{if } z < 1 \end{cases}$$

so

$$F_Z(z) = P(X^2 \leq z) = P(X^2 \leq z, X < 0) + P(X^2 \leq z, X \geq 0)$$

$$= P(-X \leq \sqrt{z}, X < 0) + P(X \leq \sqrt{z}, X \geq 0)$$

$$= \frac{1}{3} \begin{cases} 2\sqrt{z} & \text{if } 0 < z \leq 1, \\ 1 + \sqrt{z} & \text{if } 1 < z. \end{cases}$$

10. 10 points. Suppose $X$ is continuous random variable and $Y = aX + b$ where $0 < a < \infty$. Show that $Y$ is continuous and compute the pdf of $Y$ in terms of the pdf of $X$.

Solution. Suppose $y \in \mathbb{R}$. Keeping in mind that $a > 0$ we have

$$F_Y(y) = P(aX + b \leq y) = P(X \leq \frac{y - b}{a}) = F_X\left(\frac{y - b}{a}\right)$$

so that

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y - b}{a}\right) = \frac{1}{a} f_X\left(\frac{y - b}{a}\right).$$

11. 15 points. Suppose $X$ is uniform on $(0, 2)$, $Y$ is uniform on $(1, 2)$ and $X$ and $Y$ are independent. Compute the cdf of $X/Y$.

Solution. Let $Z = X/Y$. Then

$$\text{rng } X = \{x : x \in \text{rng } X\} = \{x : x \in (0, 2)\} = (0, 2).$$

Thus $F_Z(z) = 0$ if $z \leq 0$ and $F_Z(z) = 1$ if $z \geq 2$.

Suppose $0 < z < 2$. Let $B$ be the boundary of $(0, 2) \times (1, 2)$; thus $B$ is the union of the four segments $[0, 2] \times \{1\}$, $[0, 2] \times \{2\}$, $\{0\} \times [1, 2]$, $\{1\} \times [1, 2]$.

The key point is that the line $y = x/z$ meets $B$ in the points $(z, 1)$ and $(2z, 2)$ if $0 < z < 1$ and in the points $(z, 1)$ and $(2, 2/z)$ in cases $1 \leq z < 2$. Thus

$$F_Z(z) = P(Y \geq X/z) = \frac{1}{2} \begin{cases} 2 - \frac{1}{2}(2 - z)(2 - z) - 1 & \text{if } 1 < z < 2, \\ z + \frac{1}{2} & \text{if } 0 < z \leq 1. \end{cases}$$

12. 20 points. Suppose $(X, Y, Z)$ is uniformly distributed on $\{(x, y, z) \in \mathbb{R}^3 : 0 < x < y < z < 1\}$. Compute $E(Z)$. (Suggestion: I would slice by $z$.)
Solution. Let
\[ T = \{(x, y, z) : 0 < x < y < z < 1\} \]
and, for each \( z \in \mathbb{R} \), let
\[ T_z = \{(x, y) : (x, y, z) \in T\} = \begin{cases} \{(x, y) : 0 < x < z \text{ and } x < y < z\} & \text{if } 0 < z < 1, \\ \emptyset & \text{else.} \end{cases} \]

Note that
\[ \text{Area}(T_z) = \begin{cases} \frac{z^2}{2} & \text{if } 0 < z < 1, \\ 0 & \text{else.} \end{cases} \]

This gives
\[ \text{Volume}(T) = \int_0^1 \frac{z^2}{2} \, dz = \frac{1}{6} \]
which in turn gives
\[ f_{X,Y,Z}(x, y, z) = \begin{cases} 6 & \text{if } (x, y, z) \in T, \\ 0 & \text{else.} \end{cases} \]

Finally, we have
\[
E(Z) = \int \int \int_{\mathbb{R}^3} zf_{X,Y,Z}(x, y, z) \, dx \, dy \, dz \\
= 6 \int_0^1 \left( \int \int_{T_z} z \, dx \, dy \right) \, dz \\
= 6 \int_0^1 z \text{Area}(T_z) \, dz \\
= 6 \int_0^1 z \frac{z^2}{2} \, dz \\
= \frac{3}{4}.
\]