Test Two    Mathematics 135.01    Fall 2007

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

I have neither given nor received aid in the completion of this test.

Signature:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>

The average was 90.5 and the standard deviation was 28.1.

1. 1 pts. Suppose $X$ is normal with mean 3 and variance 4. Determine $a, b$ such that $aX + b$ is normal with mean 5 and variance 9.

Suppose $0 < a < \infty$ and $b \in \mathbb{R}$. We have

$$5 = E(aX + b) = aE(X) + b = 3a + b$$

and

$$9 = \text{Var}(aX + b) = a^2 \text{Var}(X) = 4a^2$$

so $a = 3/2$ and $b = 1/2$. Note that this works for not matter what the distribution of $X$ is.

2. 10 pts. Suppose $X$ is exponential distributed with parameter $\lambda = 2$. Calculate the pdf of $\sqrt{X}$.
Let \( Y = \sqrt{X} \) and note that range of \( Y \) is \((0, \infty)\). For \( 0 < y < \infty \) we have
\[
F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = 1 - e^{-2y^2}.
\]
Differentiating, we obtain
\[
f_Y(y) = \begin{cases} 
0 & \text{if } -\infty < y \leq 0, \\
4ye^{-2y^2} & \text{if } 0 < y < \infty.
\end{cases}
\]

3. 20 pts. Suppose \( X \) uniform on \((-1, 2)\). Calculate the pdf of \( X^2 + 1 \).

Let \( Y = X^2 \). The range of \( Y \) is \([1, 5)\). For \( 1 \leq y \leq 2 \) we have
\[
F_Y(y) = P(Y \leq y) = P(X^2 + 1 \leq y)
\]
\[
= P(-\sqrt{y-1} \leq X \leq \sqrt{y-1})
\]
\[
= \frac{2\sqrt{y-1}}{3}
\]
and if \( 2 < y < 5 \) we have
\[
F_Y(y) = P(Y \leq y)
\]
\[
= P(X^2 + 1 \leq y) = P(-1 \leq X \leq \sqrt{y-1})
\]
\[
= \frac{1 + \sqrt{y-1}}{3}.
\]
Differentiating, we obtain
\[
f_Y(y) = \begin{cases} 
0 & \text{if } y \leq 1, \\
\frac{1}{3\sqrt{y-1}} & \text{if } 1 \leq y \leq 2, \\
\frac{1}{\sqrt{y-1}} & \text{if } 2 < y < 5, \\
0 & \text{if } 5 \leq y.
\end{cases}
\]

4. 20 pts. Suppose \((X, Y)\) is uniformly distributed on \((0, 1) \times (0, 2)\), \( A \) is a random variable such that
\[
P(A = a) = \begin{cases} 
\frac{1}{3} & \text{if } a = 0, \\
\frac{2}{3} & \text{if } a = 1
\end{cases}
\]
and
\[
Z = \begin{cases} 
X & \text{if } A = 0, \\
Y & \text{if } A = 1.
\end{cases}
\]
Calculate the cdf of \( Z \) and determined if \( Z \) is continuous. Calculate the expectation of \( Z \). (Hint: Condition on \( A \). Consider \( \{Z \leq z\} \) for \( z \leq 0, 0 < z < 1 \) \( 1 \leq z < 2, 2 \leq z \).
Suppose $0 < z < 1$. Then
\[ F_Z(z) = P(Z \leq z) = P(Z \leq z|A = 0)P(A = 0) + P(Z \leq z|A = 1)P(A = 1) \]
\[ = P(X < z)\frac{1}{4} + P(Y < z)\frac{3}{4} \]
\[ = \frac{z}{4} + \frac{3z}{8}. \]

Suppose $1 \leq z < 2$. Then
\[ F_Z(z) = P(Z \leq z) = P(Z \leq z|A = 0)P(A = 0) + P(Z \leq z|A = 1)P(A = 1) \]
\[ = \frac{1}{4} + P(Y < z)\frac{3}{4} \]
\[ = \frac{1}{4} + \frac{3z}{8}. \]

Thus
\[ F_Z(z) = \begin{cases} 
0 & \text{if } z \leq 0, \\
\frac{z}{4} + \frac{3z}{8} & \text{if } 0 < z < 1, \\
\frac{1}{4} + \frac{3z}{8} & \text{if } 1 \leq z < 2, \\
1 & \text{if } 2 \leq z.
\end{cases} \]

Since $F_Z$ is continuous we find that $Z$ is continuous and
\[ f_Z(z) = \begin{cases} 
0 & \text{if } z \leq 0, \\
\frac{3}{8} & \text{if } 0 < z < 1, \\
\frac{3}{8} & \text{if } 1 \leq z < 2, \\
0 & \text{if } 2 \leq z.
\end{cases} \]

so
\[ E(Z) = \frac{5}{8} \int_0^1 zdz + \frac{3}{8} \int_1^2 zdz = \frac{7}{8}. \]

5. 10 pts. Suppose $(X, Y)$ is uniformly distributed on $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < x^2 < 1\}$. Calculate $E(XY)$. (For this one I want a number for answer which means you have to do whatever integrals are necessary.)

The area of $\Omega$ is
\[ \int_{-1}^1 \left( \int_0^{x^2} dy \right) dx = \frac{2}{3} \]
so
\[ f_{X,Y}(x, y) = \begin{cases} 
\frac{3}{2} & \text{if } (x, y) \in \Omega, \\
0 & \text{if } (x, y) \notin \Omega.
\end{cases} \]
Thus

\[ E(XY) = \int \int \Omega xy f_{X,Y} \, dx \, dy = \frac{3}{2} \int_{-1}^{1} \left( \int_{0}^{x^2} xy \, dy \right) \, dx = 0. \]

6. 10 pts. Suppose \(W_1, W_2, \ldots, W_n, \ldots\) are independent continuous random variables each of which is exponentially distributed with parameter \(\lambda = 2\). For each \(n = 1, 2, \ldots\) let \(T_n = \sum_{m=1}^{n} W_m\) and let

\[ N = \# \{n : 3 < T_n < 13\}. \]

Determine \(P(N = n), n = 0, 1, 2, \ldots\) (This is easy given what’s in the book and what Prof. Huber did in class. Or look in the book in the right place.)

\(N\) is Poisson with parameter \((13 - 3)\lambda = 26\) so

\[ P(N = n) = \begin{cases} e^{-26/26n} \frac{(26)^n}{n!} & \text{for } n = 0, 1, 2, \ldots, \\ 0 & \text{else}. \end{cases} \]

7. 20 pts. Suppose \((X, Y, Z)\) is uniformly distributed on \((0, 1) \times (0, 1) \times (0, 1)\). Calculate \(P(X < \min\{Y, Z\})\).

Let \(A = \{(x, y, z) \in (0, 1)^3 : x < \min\{y, z\}\}\). If \(0 < x < 1\) then the area of

\[ \{(y, z) \in (0, 1)^2 : x < \min\{y, z\}\} = \{(y, z) \in (0, 1)^2 : x < y \text{ and } x < z\} \]

is \((1 - x)^2\) so the desired probability is

\[ \text{volume}(A) = \int_{0}^{1} (1 - x)^2 \, dx = \frac{1}{3}. \]

8. 10 pts. A certain machine has five components \(C_i, i = 1, \ldots, 5\) whose failure times are exponentially distributed with parameters \(\lambda_i, i = 1, \ldots, 5\), respectively. The machine functions if \(C_1, C_2, C_3\) function or if \(C_3, C_4, C_5\) function. Calculate the expected time of failure of the machine.

Let \(Z_1 = \min\{C_1, C_2\}\), let \(Z_2 = \min\{C_4, C_5\}\), let \(Z_3 = \max\{Z_1, Z_2\}\) and let \(Z = \min\{C_3, Z_3\}\). So \(Z\) is the time of failure of the machine. Suppose
$0 < z < \infty$. Note that $Z_1$ and $Z_2$ are exponential with parameters $\lambda_1 + \lambda_2$ and $\lambda_4 + \lambda_5$. Then

\[
P(Z > z) = P(C_3 > z)P(Z_3 > z) \\
= P(C_3 > z)(1 - P(Z_3 \leq z)) \\
= P(C_3 > z)(1 - P(Z_1 \leq z)P(Z_2 \leq z) \\
= e^{-\lambda_3 z}(1 - (1 - e^{-(\lambda_1 + \lambda_2)z})(1 - e^{-(\lambda_4 + \lambda_5)z})) \\
= e^{-(\lambda_1 + \lambda_2 + \lambda_3)z} + e^{-(\lambda_1 + \lambda_4 + \lambda_5)z} - e^{-(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)z}
\]

so

\[
E(Z) = \int_0^\infty P(Z > z) \, dz \\
= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_4 + \lambda_5} - \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5}.
\]

**9. 20 pts.** I have a machine that makes widgets. Let $E_i$ be the event that the $i$th widget produced is acceptable. Assume that $P(E_i) = p$ for all $i = 1, 2, 3, \ldots$ and some $p \in (0, 1)$; assume also that the events $E_i$, $i = 1, 2, 3, \ldots$ are independent.

Suppose $N$ is a large integer and $0 < w < 1$. How many widgets does the machine have to produce so that I can be $100w\%$ sure that I get at least $N$ acceptable widgets? (I want you to use the normal approximation; your answer will depend on that $z$ such that $\Phi(z) = w$.)

For each $i = 1, 2, \ldots$ let

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{th widget is acceptable,} \\
0 & \text{if the } i\text{th widget is unacceptable.}
\end{cases}
\]

For each $n = 1, 2, \ldots$ let $S_n = \sum_{i=1}^n X_i$. Thus $S_n$ is number of acceptable widgets out of a lot of $n$ widgets. Let $q = 1 - p$ and suppose $Z$ is standard normal. Then

\[
w < P(S_n \geq N) \\
= P \left( \frac{N - np}{\sqrt{npq}} \leq \frac{S_n - np}{\sqrt{npq}} \right) \\
\approx 1 - \Phi \left( \frac{N - np}{\sqrt{npq}} \right) \\
= \Phi \left( \frac{np - N}{\sqrt{npq}} \right).
\]
Now

\[ w < \Phi \left( \frac{np - N}{\sqrt{npq}} \right) \]
\[ \iff w < \Phi^{-1}(w) < \frac{np - N}{\sqrt{npq}} \]
\[ z < \frac{np - N}{\sqrt{npq}} \]
\[ \iff \frac{np - N}{\sqrt{npq}} > z \]
\[ np - \sqrt{npq}z - N > 0 \]
\[ \iff \sqrt{n} > z\sqrt{pq} + \sqrt{z^2pq + 4pN} \]
\[ \iff n > \frac{N}{p} + \frac{z}{2p} \left( qz + \sqrt{q(z^2q + 4N)} \right). \]

10. **20 pts.** Suppose \((X, Y)\) is uniform on \((0,1) \times (0,1)\). Show that \(XY\) is continuous and calculate it’s pdf. (This is a bit tricky.)

We could use the change of variables formula to calculate \(f_{X,XY}\) and then use the fact that

\[ f_{XY}(z) = \int_{-\infty}^{\infty} f_{X,XY}(x, z) \, dx. \]

Instead, we’ll do it from first principles by calculating the cdf of \(XY\) and then differentiating.

Note that the range of \(Z = XY\) is \((0,1)\). If \(0 < z < 1\) then

\[ F_Z(z) = P(Z \leq z) \]
\[ = P(XY \leq z) \]
\[ = \text{area}(\{(x, y) \in (0,1) \times (0,1) : xy \leq z\}) \]
\[ = z + \int_{z}^{1} \frac{z}{x} \, dx \quad \text{Draw a picture!} \]
\[ = z + z \ln x \Bigg|_{x=1}^{x=z} \]
\[ = z(1 - \ln z). \]

Differentiating, we obtain

\[ F_Z(z) = \begin{cases} 
0 & \text{if } z \leq 0; \\
- \ln z & \text{if } 0 < z < 1; \\
1 & \text{if } 1 \leq z. 
\end{cases} \]
11. pts. Suppose $W_1, W_2, \ldots, W_n, \ldots$ are independent continuous random variables each of which has the same distribution. For each $n = 1, 2, \ldots$ let $T_n = \sum_{m=1}^{n} W_m$. Suppose $m$ are positive integers and $0 < a < b < \infty$. Calculate $P(bT_m < aT_{2m})$.

Suppose $m$ is a positive integer and $0 < a < b < \infty$. Calculate $P(bT_m < aT_{2m})$.

We have

$$\{bT_m < aT_{2m}\} = \left\{ T_m < \frac{a}{b-a}(T_{2m} - T_m) \right\}$$

and note that $T_m$ and $T_{2m} - T_m$ are independent random variables with the same distribution.

So suppose $0 < c < \infty$ and $X, Y$ are continuous independent random variables with the same cdf $F$ and pdf $f$. Then

$$P(X < cY) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{cx} f(x)f(y) \, dy \right) \, dx$$

$$= \int_{-\infty}^{\infty} f(x)F(cx) \, dx$$

and this is as far as you can take it. In case $c = 1$ we have

$$\int_{-\infty}^{\infty} f(x)F(x) \, dx = \int_{-\infty}^{\infty} \frac{1}{2} \frac{d}{dx}F(x)^2 \, dx = \frac{1}{2}.$$  

(Ok, I admit it, I thought you could do the above in the same way even when $c \neq 1$.)