The Poisson process.

Let

\[ I_1, I_2, \ldots, I_m, \ldots \]

be a sequence of independent identically distributed continuous random variables. Let \( T_0 = 0 \) and, for each positive integer \( m \), let

\[ T_m = \sum_{i=0}^{m} I_i. \]

Evidently,

\[ 0 = T_0 < T_1 < \cdots < T_m < \cdots. \]

For nonnegative integers \( m, n \) with \( m \leq n \) let

\[ T_{m,n} = \sum_{m<i \leq n} I_i; \]

note that

\[ T_n = T_m + T_{m,n}; \]

and that \( T_{m,n} \) and \( T_{m-n} \) have the same distribution, which is to say that \( f_{T_{m,n}} = f_{T_{m-n}} \).

Let \( F : \mathbb{R} \to [0,1] \) be such that

\[ F(t) = P(I_m \leq t) \quad \text{whenever } t \in \mathbb{R} \text{ and } m = 1,2,\ldots \]

and let

\[ f = F'. \]

Let \( f_1 = f \) and, for each \( m = 2,3,\ldots \) let

\[ f_m = f \ast \cdots \ast f. \]

For each \( m = 1,2,\ldots \) let

\[ F_m(t) = \int_{-\infty}^{t} f_m(\tau) \, d\tau \quad \text{whenever } t \in \mathbb{R}. \]

**Theorem.** We have

\[ F_{T_m} = F_m, \quad \text{and} \quad f_{T_m} = f_m \quad m = 1,2,\ldots. \]

**Proof.** This was shown earlier. \( \square \)

**Theorem.** Suppose \( k \) and \( m_1, \ldots, m_k \) are positive integers,

\[ m_k > \cdots > m_1, \]

\( t_1, \ldots, t_k \) are positive real numbers and

\[ t_k > \cdots > t_1. \]

Then

\[ f_{T_{m_k},\ldots,T_{m_1}}(t_k,\ldots,t_1) = f_{m_k-m_{k-1}}(t_k-t_{k-1}) \cdots f_2(t_2-t_1)f_1(t_1). \]
Proof. This is a good exercise in conditioning.
The key point is the following. Suppose \( j = 1, \ldots, k \). For any \( u \in \mathbb{R} \) we have
\[
P(T_{m_j} \leq u | T_{m_{j-1}} = t_{j-1}, \ldots, T_{m_1} = t_1) = \frac{P(T_{m_{j-1}} + T_{m_j} \leq u | T_{m_{j-1}} = t_{j-1}, \ldots, T_{m_1} = t_1)}{P(T_{m_{j-1}} = t_{j-1})}
\]
which implies that
\[
f_{T_{m_j} | T_{m_{j-1}}, \ldots, T_1}(t_j | t_{j-1}, \ldots, t_1) = f_{T_{m_{j-1}}}(t_j - t_{j-1}).
\]
It follows that
\[
f_{T_{m_k}, \ldots, T_{m_1}}(t_k, \ldots, t_1) = f_{T_{m_k} | T_{m_{k-1}}, \ldots, T_{m_1}}(t_k | t_{k-1}, \ldots, t_1) \cdots f_{T_{m_2} | T_{m_1}}(t_2 | t_1) f_{T_{m_1}}(t_1)
\]
\[
= f_{T_{m_k-m_{k-1}}} - f_{T_1}(t_1).
\]
For each \( t \in (0, \infty) \) we let \( N_t \) be the nonnegative integer random variable such that
\[
\{N_t = n\} = \{T_n \leq t < T_{n+1}\} \quad \text{for any nonnegative integer } n.
\]

Theorem. For each \( t \in (0, \infty) \) we have
\[
P(N_t = n) = \int_0^t \left( \int_0^\infty f(t_{n+1} - t_n) \, dt_{n+1} \right) f_n(t_n) \, dt_n.
\]

Proof. We have
\[
P(N_t = n) = P(T_n \leq t, T_{n+1} > t) = \int \int_{t_n \leq t < t_{n+1}} f_{T_{n+1}, T_n}(t_{n+1}, t_n) \, dt_{n+1} \, dt_n.
\]
Now apply (3). \( \square \)

Theorem. Suppose \( s \) and \( t \) are positive real numbers and \( m \) and \( n \) are nonnegative integers. Then
\[
P(N_{s+t} = m + n, N_s = m) = \int_0^s \left( \int_0^{s+t} \left( \int_0^{s+t} g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) \, dt_{m+n+1} \right) \, dt_{m+n} \right) \, dt_m
\]
where we have set
\[
g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) = f(t_{m+n+1} - t_{m+n}) f_{n-1}(t_{m+n} - t_{m+1}) f(t_{m+1} - t_m) f_m(t_m)
\]
for \( 0 < t_m < t_{m+1} < t_{m+n} < t_{m+n+1} \).
Proof. Since
\[ P(N_{s+t} = m + n, N_s = m) = P(T_m \leq s < T_{m+1}, T_{m+n} \leq s + t < T_{m+n+1}) \]
the desired probability is the integral over
\[ \{(t_{m+n+1}, t_{m+n}, t_{m+1}, t_m) : 0 < t_m \leq s < t_{m+1}, t_{m+n} \leq s + t < t_{m+n+1} < \infty \} \]
of the joint density
\[ f_{T_{m+n+1}, T_{m+n}, T_{m+1}, T_m}(t_{m+n+1}, t_{m+n}, t_{m+1}, t_m). \]
Now apply (3). □

The Poisson process. Now suppose \( 0 < \lambda < \infty \) and
\[ P(I_m > t) = e^{-\lambda t} \quad \text{whenever} \quad 0 < t < \infty \text{ and } m = 1, 2, \ldots. \]
That is, \( I_m, m = 1, 2, \ldots \) is exponentially distributed with parameter \( \lambda \).

Theorem. For any \( m = 1, 2, \ldots \) we have
\[ f_m(t) = \begin{cases} 0 & \text{if } t < 0, \\ \lambda^m e^{-\lambda t} \frac{t^{m-1}}{(m-1)!} & \text{if } t > 0. \end{cases} \tag{5} \]

Proof. We induct on \( n \). (4) holds by definition if \( m = 1 \).
Suppose (5) holds for some positive integer \( k \) and \( t > 0 \) then
\[ f_{k+1}(t) = f * f_k(t) = \int_0^t e^{-\lambda(t-\tau)} e^{-\lambda \tau} \frac{\tau^{m-1}}{(m-1)!} d\tau = e^{-\lambda t} \int_0^t \frac{\tau^{m-1}}{(m-1)!} d\tau = e^{-\lambda t} \int_0^t \frac{t^m}{m!}. \]
□

Theorem. For any \( t \in (0, \infty) \) and any nonnegative integer \( n \) we have
\[ P(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{6} \]

Remark. Thus \( N_t \) has the Poisson distribution with parameter \( \lambda t \).

Proof. Suppose \( n \) is a nonnegative integer. By (3) and (5)
\[ P(N_t = n) = \int_0^t \left( \int_t^\infty e^{-\lambda(t_{n+1} - t_n)} dt_{n+1} \right) \lambda^n e^{-\lambda t_n} \frac{t_n^{n-1}}{(n-1)!} \ dt_n = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \]
□

Theorem. Suppose \( s, t \in (0, \infty) \) and \( m, n \) are nonnegative integers. Then
\[ P(N_{s+t} = m + n, N_s = m) = P(N_s = m) P(N_t = n). \tag{7} \]

Proof. This will follow from (5) and (4).
Suppose $n > 1$. If $g$ is as in (4) we have from (5) that
\[ g(t_{m+n+1}, t_{m+n}, t_{m+1}, t_m) = \lambda^{m+n+1} e^{-\lambda t_{m+n+1}} \frac{(t_{m+n} - t_{m+1})^{n-2}}{(n-2)!} \frac{t_m^{m-1}}{(m-1)!} \]
whenever $0 < t_m < t_{m+1} < t_{m+n+1} < t_{m+n+1} < \infty$. Then
\[
\int_0^s \left( \int_0^s \left( \int_0^{t_{m+n+1}} g(t_m, t_{m+1}, t_m+n, t_{m+n+1}) dt_{m+n+1} \right) dt_{m+n} \right) dt_{m+1} dt_m \n\]
\[
\lambda^{m+n+1} \left( \int_{s+t}^{\infty} e^{-\lambda t_{m+n+1}} dt_{m+n+1} \right) \left( \int_{t_{m+n+1}}^{s+t} \frac{(t_{m+n} - t_{m+1})^{n-2}}{(n-2)!} dt_{m+n} \right) \left( \int_0^t \frac{t_m^{m-1}}{(m-1)!} dt_m \right) \n\]
\[
= \lambda^{m+n+1} e^{-\lambda(s+t)} \frac{s^n}{n!} \frac{s^m}{m!}. \]

We leave it to the reader to use similar techniques to handle the case $n = 0$ or $n = 1$.

**Corollary.** Suppose $s$ and $t$ are positive real numbers. Then $N_{s+t} - N_s$ is independent of $N_s$ and has the Poisson distribution with parameter $\lambda t$.

**Proof.** We have
\[
P(N_{s+t} - N_s = n, N_s = m) = P(N_{s+t} = m+n, N_s = m) = P(N_s = m) P(N_t = n) \]
for any nonnegative integers $m$ and $n$. Summing over $m$ in (1) we infer that $N_{s+t} - N_s$ has the same distribution as $N_t$. Substituting $P(N_t = n) = (N_{s+t} - N_s = n)$ in the right hand side of (1) we infer that $N_{s+t} - N_s$ is independent of $N_s$. \( \square \)

**Remark.** Suppose $m$ is an integer not less than 2 and $0 < t_1 < t_2 < \cdots < t_m < \infty$. Applying the preceding Corollary repeatedly we infer that
\[ N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_m} - N_{t_{m-1}} \]
are independent with Poisson distributions with parameters
\[ \lambda t_1, \lambda(t_2 - t_1), \ldots, \lambda(t_m - t_{m-1}), \]
respectively.

We say the Poisson process has **independent increments**.