The multinomial distribution.

Let $n$ and $r$ be positive integers and let 

$$p_1, \ldots, p_r$$

be such that 

$$0 \leq p_\alpha \leq 1 \quad \text{and} \quad \sum_{\alpha=1}^{r} p_\alpha = 1.$$

Let 

$$M(n, r)$$

be the set of $r$-tuples $(n_1, \ldots, n_r)$ of nonnegative integers such that $\sum_{\alpha=1}^{r} n_\alpha = n$. The discrete random vector $(N_1, \ldots, N_r)$ is said to have the $r$-nomial (multinomial in the book) distribution with parameters $n$ and $p_1, \ldots, p_r$ if 

$$p_{N_1, \ldots, N_r}(n_1, \ldots, n_r) = \begin{cases} \binom{n}{n_1 \ldots n_r} p_1^{n_1} \cdots p_r^{n_r} & \text{if } (n_1, \ldots, n_r) \in M(n, r), \\ 0 & \text{else}. \end{cases}$$

Note that 

$$1 = (p_1 + \cdots + p_r)^n = \sum_{(n_1, \ldots, n_r) \in M(n, r)} \binom{n}{n_1 \ldots n_r} p_1^{n_1} \cdots p_r^{n_r}$$

by the multinomial theorem.

**Example.** An urn contain 3 red balls, 4 white balls and 5 blue balls. A ball is drawn from the urn 10 times with replacement. Let $R, W$ and $B$ be the number of red, white and blue balls drawn, respectively. Then, as we shall see below, $(R, W, B)$ will have the 3-nomial distribution with parameters $n = 10$ and 

$$p_1 = \frac{3}{3+4+5}, \quad p_2 = \frac{4}{3+4+5}, \quad p_2 = \frac{5}{3+4+5}.$$ 

Note that the range of $(R, B, W)$ is the set of 3-tuples $(n_r, n_w, n_b)$ of nonnegative integers such that $n_r + n_w + n_b = 10$.

Here is how such a random vector arises. Let 

$$O = \{o_1, \ldots, o_r\}$$

be a set containing exactly $r$ elements; $O$ is the set of *outcomes*. Let 

$$X_i, \ i = 1, \ldots, n,$$

be independent identically distributed random variables with the same range $O$ such that 

$$P(X_i = o_\alpha) = p_\alpha \quad \text{whenever } i \in \{1, \ldots, n\} \text{ and } \alpha \in \{1, \ldots, r\}.$$ 

For each $\alpha \in \{1, \ldots, r\}$ let 

$$N_\alpha = \sum_{i=1}^{n} 1_{\{X_i = o_\alpha\}};$$

thus $N_\alpha$ is the number of times the $\alpha$-th outcome $o_\alpha$ occurs in $n$ tries. Evidently, 

$$\sum_{\alpha=1}^{r} N_\alpha = n.$$
Note that $N_n$, being the sum of $n$ independent Bernoulli variables with parameter $p$, is binomial with parameters $n = n$ and $p = p_n$. We let

$$A(n, r)$$

be the family of ordered $r$-tuples $(A_1, \ldots, A_r)$ of subsets of $\{1, \ldots, n\}$ such that $A_i \cap A_j = \emptyset$ and, for each $A \in A(n, r)$, we let

$$E_A = \cap_{\alpha=1}^r \cap_{i \in A_\alpha} \{X_i = o\}$$

and observe that, by the independence of the parameters $n$

$$P(E_A) = \prod_{\alpha=1}^r p_{A_\alpha}. \tag{2}$$

If $(n_1, \ldots, n_r)$ is an $r$-tuple of nonnegative integers summing to $n$ we find that

$$P(N_1 = n_1, \ldots, N_r = n_r) = P(\cup_{A \in A(n, r), |A_\alpha| = n_\alpha, \alpha = 1, \ldots, r} E_A) = \binom{n}{n_1 \ldots n_r} p_{n_1} \ldots p_{n_r}$$

so (1) holds.

**Theorem.** Suppose $q$ is an integer, $1 < q < r$, $\hat{p} = \sum_{\alpha=1}^q p_\alpha$ and

$$\hat{p}_\alpha = \frac{p_\alpha}{\hat{p}} \text{ whenever } \alpha = 1, \ldots, q.$$

Then

$$P(N_1 = n_1, \ldots, N_q = n_q|N_{q+1} = n_{q+1}, \ldots, N_r = n_r) = \left(\sum_{\alpha=1}^q n_\alpha\right) \hat{p}_{n_1} \ldots \hat{p}_{n_q}$$

whenever $(n_1, \ldots, n_r) \in M(n, r).

**Remark.** This says that the conditional distribution of $(N_1, \ldots, N_q)$ given $\{N_{q+1} = n_{q+1}, \ldots, N_r = n_r\}$ is $q$-nomial with parameters $n - \sum_{\alpha=q+1}^r n_\alpha$ and $\hat{p}_{q+1}, \ldots, \hat{p}_r$ whenever $(n_{q+1}, \ldots, n_r)$ is a $(r-q)$-tuple of nonnegative integers whose sum does not exceed $n$. In particular, if $\alpha \in \{1, \ldots, q\}$ we find that the conditional distribution of $N_\alpha$ given $\{N_{q+1} = n_{q+1}, \ldots, N_r = n_r\}$ is binomial with parameters $n = n - \sum_{\alpha=q+1}^r n_\alpha$ and $p = p_\alpha$.

**Proof.** Suppose $(n_1, \ldots, n_r) \in M(n, r)$. Let

$$E = \cap_{\alpha=1}^q \{N_\alpha = n_\alpha\} \text{ and let } F = \cap_{\beta=q+1}^r \{N_\beta = n_\beta\}.$$ 

Let $\hat{n} = \sum_{\alpha=1}^q n_\alpha$.

Let $B$ be the family of $q$-tuples $(B_1, \ldots, B_q)$ of subsets of $\{1, \ldots, n\}$ such that $|B_\alpha| = n_\alpha$ whenever $\alpha \in \{1, \ldots, q\}$ and $B_\alpha \cap B_\beta = \emptyset$ whenever $\alpha, \beta \in \{1, \ldots, q\}$ and $\alpha \neq \beta$. For each $B \in B$ let

$$E_B = \cap_{\alpha=1}^q \cap_{i \in B_\alpha} \{X_i = o\}.$$ 

Let $B'$ be the set of those $(B_1, \ldots, B_q) \in B$ such that $B_\alpha \subset \{1, \ldots, \hat{n}\}$ whenever $\alpha \in \{1, \ldots, q\}$.

Let $C$ be the family of $(r-q)$-tuples $(C_{q+1}, \ldots, C_r)$ of subsets of $\{1, \ldots, n\}$ such that $|C_\beta| = n_\beta$ whenever $\beta \in \{q+1, \ldots, r\}$ and $C_\beta \cap C_\gamma = \emptyset$ whenever $\beta, \gamma \in \{q+1, \ldots, r\}$ and $\beta \neq \gamma$. Let $\hat{C}$ be that member of $B$ such that

$$\hat{C}_\beta = \left\{\sum_{\gamma=1}^{\beta-1} n_\gamma + 1, \ldots, \sum_{\gamma=1}^{\beta} n_\gamma\right\}, \beta = q+1, \ldots, r.$$ 

For each $B \in B$ let

$$F_B = \cap_{\beta=q+1}^r \cap_{i \in C_\beta} \{X_i = o_\beta\}.$$
One may easily verify that

\[ P(F_C) = P(F_D) \quad \text{and} \quad P(E|F_C) = P(E|F_D) \quad \text{whenever} \ C, D \in \mathbb{C}. \]

By an earlier result we have that

\[ P(E|F) = P(E|F_C). \]

Moreover,

\[
P(E|F_C) = \sum_{(B_1, \ldots, B_q) \in B} P(E_B | F_C)
\]

\[
= \sum_{(B_1, \ldots, B_q) \in B} \frac{P(E_B \cap F_C)}{P(F_C)}
\]

\[
= \sum_{(B_1, \ldots, B_q) \in B'} \frac{P(E_B \cap F_C)}{P(F_C)}
\]

\[
= \sum_{(B_1, \ldots, B_q) \in B'} \frac{p_1^{n_1} \cdots p_q^{n_q} p_{q+1}^{n_{q+1}} \cdots p_r^{n_r}}{\tilde{p}^q p_{q+1}^{n_{q+1}} \cdots p_r^{n_r}}
\]

\[
= \left( \frac{\tilde{n}}{n_1 \cdots n_q} \right) \tilde{p}^{n_1} \cdots \tilde{p}^{n_q}.
\]

\[ \square \]