A Fundamental Formula. Suppose

(i) \( X \) is a continuous random \( n \)-vector;

(ii) \( g \) is a continuously differentiable real valued function on the range of \( X \);

(iii) \( Z = g(X) \);

(iv) \( P(X \in B) = 0 \) where \( B = \{ x \in \text{rng} \; X : \nabla g(x) = 0 \} \).

Then \( Z \) is continuous and

\[
f_Z(z) = \int_{\{ x : g(x) = z \}} \frac{f_X(x)}{|\nabla g(x)|} \, dx.
\]

We may extend this formula as follows. Suppose

\( A \subset \mathbb{R}^n, \; g : A \to \mathbb{R}, \; \psi : A \to \mathbb{R}, \; z \in \mathbb{R} \)

are such that \( g \) is continuously differentiable, \( \psi \) is continuous and

\[
\nabla g(x) \neq 0 \quad \text{whenever} \quad x \in A, g(x) = z \quad \text{and} \quad \psi(x) \neq 0.
\]

Suppose

\( B \subset \mathbb{R}^{n-1}, \; f : B \to \mathbb{R} \) is continuously differentiable, \( \{ x : g(x) = z \} = f. \)

Then

\[
\int_{\{ x : g(x) = z \}} \psi(x)/|\nabla g(x)| \, dx = \int_A \psi(u, f(u))/|\frac{\partial g}{\partial x_n}(u, f(u))| \, du.
\]

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