Linear transformations and Gaussian random vectors.

Remember, n-vectors are the same as $n \times 1$ matrices.

Let $\mathbf{X}$ a random n-vector. We let

$$\mathrm{E}(\mathbf{X})$$

be the n-vector whose i-th entry is $\mathrm{E}(X_i)$. If $\mathbf{Y}$ is a random n-vector we let

$$\mathrm{Cov}(\mathbf{X}, \mathbf{Y})$$

be the $n \times n$ matrix whose i, j entry is $\mathrm{Cov}(X_i, Y_j)$ and we let

$$\mathrm{Var}(\mathbf{X}) = \mathrm{Cov}(\mathbf{X}, \mathbf{X}).$$

We have already proved the simple

Proposition. Suppose $\mathbf{X}$ is a random n-vector, A is an $n \times n$ matrix, $\mathbf{b}$ is an n-vector and

$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}.$$

Then

$$\mathrm{E}(\mathbf{Y}) = A\mathrm{E}(\mathbf{X}) + \mathbf{b}$$

and

$$\mathrm{Var}(\mathbf{Y}) = A\mathrm{Cov}(\mathbf{X})A^T.$$

Definition. We say the random vector $\mathbf{X}$ is **standard normal** if its components $X_1, \ldots, X_n$ are independent and standard normal. Evidently, this is the case if and only if $\mathbf{X}$ is continuous and

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-n/2} e^{-|\mathbf{x}|^2/2} \qquad \text{for any } n\text{-vector } \mathbf{x}.$$

We say the random vector $\mathbf{Y}$ is **Gaussian** if

$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}$$

for some standard normal $\mathbf{X}$ and nonsingular A. Note that

$$\mathrm{E}(\mathbf{Y}) = \mathbf{b} \qquad \text{and} \qquad \mathrm{Var}(\mathbf{Y}) = AA^T.$$

Discussion. Let $\mathbf{X}$ be standard normal and suppose

$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}$$

for some nonsingular A, so that $\mathbf{Y}$ is Gaussian. Let $B = \sqrt{AA^T} = \sqrt{\mathrm{Var}(\mathbf{Y})}$; B is well defined because AA^T is symmetric positive definite. If $\mathbf{y} = A\mathbf{x} + \mathbf{b}$ we have

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-n/2} e^{-|\mathbf{x}|^2/2} |\mathbf{det}\, A|^{-1} = (2\pi)^{-n/2} |\mathbf{det}\, A|^{-1} e^{-|A^{-1}(\mathbf{y}-\mathbf{b})|^2/2}.$$

But

$$\begin{aligned} |A^{-1}(\mathbf{y}-\mathbf{b})|^2 &= (A^{-1}(\mathbf{y}-\mathbf{b}))^T A^{-1}(\mathbf{y}-\mathbf{b}) \\ &= (\mathbf{y}-\mathbf{b})^T (AA^T)^{-1}(\mathbf{y}-\mathbf{b}) \\ &= (B^{-1}(\mathbf{y}-\mathbf{b}))^T B^{-1}(\mathbf{y}-\mathbf{b}) \\ &= |B^{-1}(\mathbf{y}-\mathbf{b})|^2 \end{aligned}$$

and

$$\mathbf{det}\, A = \sqrt{\mathbf{det}\, AA^T} = \mathbf{det}\, B.$$

Thus we have the following

Theorem. Suppose $\mathbf{Y}$ is a random n-vector. Then $\mathbf{Y}$ is Gaussian if and only if $\mathrm{Var}(\mathbf{Y})$ is positive definite and

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-n/2} (\mathbf{det} \sqrt{\mathrm{Var}(\mathbf{Y})})^{-1} e^{-|(\sqrt{\mathrm{Var}(\mathbf{Y})})^{-1}(\mathbf{y} - \mathrm{E}(\mathbf{Y})|^2/2}$$

for any n-vector $\mathbf{y}$ in which case

$$\mathbf{X} = (\sqrt{\mathrm{Var}\mathbf{Y}})^{-1}(\mathbf{y} - \mathrm{E}(\mathbf{Y}))$$

is standard normal.