Linear transformations and Gaussian random vectors.

We fix a positive integer $n$.

1. Symmetric matrices; psd matrices.

When we write $x \in \mathbb{R}^n$ we mean that

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Let $\text{Sym}(n)$ be the vector space of $n \times n$ symmetric matrices. We say the $n \times n$ matrix $B$ is positive definite symmetric (psd) if $B$ is symmetric and

$$(1) \quad x^T B x > 0 \quad \text{whenever} \quad x \in \mathbb{R}^n.$$

If the $n \times n$ matrix $B$ is symmetric then (1) is equivalent to the statement that the eigenvalues of $B$ are positive. Remember, $n$-vectors are the same as $n \times 1$ matrices.

Suppose $B \in \text{Sym}(n)$. Then there are $n \times n$ matrices $P, D$ such that $P$ is orthogonal (which means $P$ is invertible and $P^{-1} = P^T$), $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and

$$B = PDP^{-1}.$$

This amounts to saying that, for each $j = 1, \ldots, n$,

$$BP_j = \lambda_j P_j$$

where $P_j$ is the $j$th column of $B$; in other words, $P_j$ is a (nonzero) eigenvector of $B$ with eigenvalue $\lambda_j$.

**Theorem 1.1.** Suppose $f : \mathbb{R} \to \mathbb{R}$. There is one and only one function, also denoted by

$$f : \text{Sym}(n) \to \text{Sym}(n)$$

(ambiguously denoted by $f$!) which preserves the matrix operations and such that

$$f(\text{diag}(\lambda_1, \ldots, \lambda_n)) = \text{diag}(f(\lambda_1), \ldots, f(\lambda_n)).$$

Moreover, if $g : \mathbb{R} \to \mathbb{R}$ we have

$$(fg)(B) = f(B)g(B) \quad \text{for} \quad B \in \text{Sym}(n).$$

**Proof.** We let

$$f(B) = P\text{diag}(f(\lambda_1), \ldots, f(\lambda_n))P^{-1}$$

where $P, D$ and $\lambda_i, i = 1, \ldots, n$ are as above. Note that $P, D$ and $\lambda_i, i = 1, \ldots, n$ are not unique; we leave it to the reader that $f(B)$ is, nonetheless, well defined. We also leave to the reader the simple task of verifying the other asserted properties of the mapping $B \to f(B)$. $\square$
2. Gaussian random vectors.

Let $X$ a random $n$-vector. Recall the definitions of
$$E(X), \quad \text{Cov}(X, Y);$$
these are, respectively, a vector in $\mathbb{R}^n$ and an $n$ by $n$ symmetric matrix.

**Theorem 2.1.** Suppose $Y$ is a random vector, $B$ is an $n$ by $n$ psd matrix and $m \in \mathbb{R}^n$. The following are equivalent:

(I) $Y = \sqrt{B}X + m$ for some standard normal $X$.

(II) $Y$ is continuous and
$$f_Y(y) = (2\pi)^{-n/2} \sqrt{\det Be} e^{-(y-m)^T B^{-1} (y-m)/2} \quad \text{for} \quad y \in \mathbb{R}^n.$$ 

If these conditions hold then
$$E(Y) = m \quad \text{and} \quad \text{Cov}(Y, Y) = B.$$ 

**Proof.** Suppose (I) holds. Then $Y$ is continuous and, if $x, y \in \mathbb{R}^n$ are such that $y = \sqrt{B}x + m$ then $x = \sqrt{B}^{-1}(y - m)$ so
$$|x|^2 = x^T x = (\sqrt{B}^{-1}(y - m))^T \sqrt{B}^{-1}(y - m) = (y - m)^T B^{-1}(y - m);$$
thus, by Change of Variables Formula for Random Vectors,
$$f_Y(y) = f_X(x) \frac{1}{\det \left( \sqrt{B}^{-1} \right)}$$
$$= (2\pi)^{-n/2} \sqrt{\det Be} e^{-|x|^2/2}$$
$$= (2\pi)^{-n/2} \sqrt{\det Be} e^{-(y-m)^T B^{-1} (y-m)/2}.$$ 

Thus (II) holds.

Suppose (II) holds. Let $X = \sqrt{B}^{-1}(Y - m)$. Then $X$ is continuous and, if $x, y$ are as above,
$$f_X(x) = f_Y(y) \frac{1}{\det \sqrt{B}^{-1}}$$
$$= (2\pi)^{-n/2} \sqrt{\det Be} e^{-(y-m)^T B^{-1} (y-m)/2} \frac{1}{\det \sqrt{B}^{-1}}$$
$$= (2\pi)^{-n/2} e^{-|x|^2/2}$$
by the Change of Variables Formula for Random Vectors. So $X$ is standard normal.

The assertion about the mean and covariance follow from straightforward homework exercises. □

**Theorem 2.2.** Suppose $Y$ is Gaussian, $A$ is a nonsingular $n$ by $n$ matrix and $b \in \mathbb{R}^n$. Then $AY + b$ is Gaussian.

**Proof.** Exercise for the reader; just turn the crank using the Change of Variables Formula for Random Vectors. □