Computing expectation by conditioning.

Let \((S, \mathcal{E}, P)\) be a probability space, let \(F \in \mathcal{E}\) be such that
\[
P(F) > 0.
\]
Suppose \(X\) is a discrete random variable. We let
\[
E(X|F)
\]
be the expectation of \(X\) with respect to the probability \(P(\cdot | F)\). It follows that
\[
E(X|F) = \sum_x x P(X = x | F).
\]
Moreover, if \(F_1, \ldots, F_n \in \mathcal{E}\) are such that \(F_i \cap F_j = \emptyset\) whenever \(i \neq j\) and \(S = \cup_{i=1}^n F_i\) then, as one may easily verify,
\[
E(X) = \sum_{i=1}^n E(X|F_i)P(F_i).
\]

Example. Let \(X_1, X_2, X_3, \ldots\) be a sequence of independent Bernoulli random variable with parameter \(p > 0\). Let \(q = 1 - 0\) and let \(G = \min\{n : X_n \neq 0\}\) so \(G\) is geometric with parameter \(p\).

Proposition. We have
\[
P(G = n|X_1 = 0) = P(G + 1 = n) \quad \text{and} \quad P(G = n|X_1 = 1) = P(1 = n) \quad \text{for any positive integer } n.
\]

Proof. We have
\[
P(G = 1|X_1 = 1) = P(X_1 = 1|X_1 = 1) = 1 = P(1 = 1)
\]
and
\[
P(G = 1|X_1 = 0) = P(X_1 = 1|X_1 = 0) = 0 = P(G + 1 = 1)
\]
so these equations hold if \(n = 1\). If \(n > 1\) we have
\[
P(G = n|X_1 = 1) = P(X_1 = 0, \ldots, X_{n-1} = 0, X_n = 1|X_1 = 1) = 0 = P(1 = n)
\]
and
\[
P(G = n|X_1 = 0) = P(X_1 = 0, \ldots, X_{n-1} = 0, X_n = 1|X_1 = 1) = P(X_1 = 0, \ldots, X_{n-1} = 0, X_n = 1, X_1 = 0)
\]
\[
= P(X_1 = 0) \cdot P(X_n = 1 | X_1 = 0) = P(X_1 = 0) \cdot P(X_n = 1 | X_1 = 0)
\]
\[
= q^{n-2} p = P(G = n - 1) = P(G + 1 = n).
\]

\(\Box\)
We infer that

\[ E(G|X_1 = 0) = E(G + 1) = E(G) + 1 \quad \text{and} \quad E(G|X_1 = 1) = E(1) = 1. \]

It follows from (2) that

\[ E(G) = E(G|X_1 = 0)P(X_1 = 0) + E(G|X_1 = 1)P(X_1 = 1) = (E(G) + 1)q + 1p \]

which gives

\[ E(G) = \frac{1}{p}. \]

We obtain from (3) that, for any positive integer \( n \),

\[ P(G^2 = n^2|X_1 = 0) = P(G = n|X_1 = 0) = P(G + 1 = n|X_1 = 0) = P((G + 1)^2 = n^2|X_1 = 0) \]

and

\[ P(G^2 = n^2|X_1 = 1) = P(G = n|X_1 = 1) = P(1 = n) = P(1 = n^2) \]

It follows from (2) that

\[ E(G^2) = E(G^2|X_1 = 0)P(X_1 = 0) + E(G|X_1 = 1)P(X_1 = 1) = (E(G^2) + 2E(G) + 1)q + 1p \]

which gives

\[ E(G) = \frac{1}{p} \quad \text{and} \quad E(G^2) = \frac{2 - p}{p^2}. \]

In particular,

\[ Var(G) = E(G^2) - E(G)^2 = \frac{q}{p^2}. \]