The vibrating string.

Suppose $0 < L < \infty$.

We start with the fundamental fact that, given a sufficiently nice

$$u : [0, L] \to\to \mathbf{R},$$

we have

$$u(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad x \in [0, L],$$

in a sense we will make precise later, where

$$b_n = \frac{2}{L} \int_0^L u(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, 3, \ldots.$$

Now suppose $0 < c < \infty$. Let

$$f, g : [0, L] \to \mathbf{R}$$

be given. We seek

$$u : [0, L \times [0, \infty) \to \mathbf{R}$$

such that

(PDE) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2};$$

(BC) $$u(0, t) = u(L, t) \quad \text{for } t \in [0, \infty).;$$

(IC) $$u(x, 0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x).$$

We solve this by assuming u has the form

(1) $$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}, \quad (t, x) \in [0, \infty) \times [0, L];$$

here

$$b_n : [0, \infty) \to \mathbf{R}, \quad n = 1, 2, 3, \ldots$$

is a sequence of functions to be determined. The motivation is that

$$[0, L] \ni x \mapsto\to \sin \frac{n\pi x}{L}$$

is an eigenfunction of

$$c^2 \frac{\partial^2}{\partial x^2}$$

(with eigenvalue $\left(-\frac{cn\pi}{l}\right)^2$) and, in view of the above, any function can be written as a linear combination, albeit infinite, of these eigenfunctions.

Proceeding formally, we obtain

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \sum_{n=1}^{\infty} \ddot{b}_n(t) \sin \frac{n\pi x}{L}$$

and

$$c^2 \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -\left(\frac{cn\pi}{L}\right)^2 b_n(t) \sin \frac{n\pi x}{L}$$

for $(x,t) \in [0,L] \times [0,\infty)$ which suggests requiring that

$$\ddot{b}_n(t) + \left(\frac{cn\pi}{L}\right)^2 b_n(t) = 0, \quad n = 1,2,3,\ldots,\ t \in [0,\infty)$$

which says

$$b_n(t) = b_n(0) \cos \frac{cn\pi t}{L} + \dot{b}_n(0) \frac{L}{c\pi n} \sin \frac{cn\pi t}{L}, \quad n = 1,2,3,\ldots,\ t \in [0,\infty).$$

Since

$$f(x) = \sum_{n=1}^{\infty} b_n(0) \sin \frac{n\pi x}{L} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \dot{b}_n(0) \sin \frac{n\pi x}{L}, \quad x \in [0,L],$$

we should have

$$b_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad \text{and} \quad \dot{b}_n(0) = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1,2,3,\ldots.$$