The vibrating string.

Suppose $0 < L < \infty$.

We start with the fundamental fact that, given a sufficiently nice

$$u : [0, L] \rightarrow \mathbb{R},$$

we have

$$u(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad x \in [0, L],$$

in a sense we will make precise later, where

$$b_n = \frac{2}{L} \int_{0}^{L} u(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots.$$ 

Now suppose $0 < c < \infty$. Let

$$f, g : [0, L] \rightarrow \mathbb{R}$$

be given. We seek

$$u : [0, L] \times [0, \infty) \rightarrow \mathbb{R}$$

such that

\begin{align*}
\text{(PDE)} & \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}; \\
\text{(BC)} & \quad u(0, t) = u(L, t) \quad \text{for } t \in [0, \infty); \\
\text{(IC)} & \quad u(x, 0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x).
\end{align*}

We solve this by assuming $u$ has the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}, \quad (t, x) \in [0, \infty) \times [0, L];$$

here

$$b_n : [0, \infty) \rightarrow \mathbb{R}, \quad n = 1, 2, 3, \ldots$$

is a sequence of functions to be determined. The motivation is that

$$[0, L] \ni x \mapsto \sin \frac{n\pi x}{L}$$

is an eigenfunction of

$$c^2 \frac{\partial^2}{\partial x^2}$$

(with eigenvalue $(-\frac{c^2 \pi^2}{L^2})$) and, in view of the above, any function can be written as a linear combination, albeit infinite, of these eigenfunctions.
Proceeding formally, we obtain
\[ \frac{\partial^2 u}{\partial t^2}(t, x) = \sum_{n=1}^{\infty} \ddot{b}_n(t) \sin \frac{n\pi x}{L} \]
and
\[ c^2 \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -\left( \frac{c n \pi}{L} \right)^2 b_n(t) \sin \frac{n\pi x}{L} \]
for \((x, t) \in [0, L] \times [0, \infty)\) which suggests requiring that
\[ \ddot{b}_n(t) + \left( \frac{c n \pi}{L} \right)^2 b_n(t) = 0, \quad n = 1, 2, 3, \ldots, \quad t \in [0, \infty) \]
which says
\[ b_n(t) = b_n(0) \cos \frac{c n \pi t}{L} + \dot{b}_n(0) \frac{L}{c n \pi} \sin \frac{c n \pi t}{L}, \quad n = 1, 2, 3, \ldots, \quad t \in [0, \infty). \]

Since
\[ f(x) = \sum_{n=1}^{\infty} b_n(0) \sin \frac{n\pi x}{L} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \dot{b}_n(0) \sin \frac{n\pi x}{L}, \quad x \in [0, L], \]
we should have
\[ b_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \quad \text{and} \quad \dot{b}_n(0) = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots. \]