A example of nonuniqueness.

Fix a real number p with $1 < p < 2$. Let

$$f(t,x) = p|x|^{\frac{p-1}{p}} \quad \text{for } (t,x) \in \mathbf{R} \times \mathbf{R}.$$

Note that f is continuous but that f is *not* regular.

Consider the **initial value problem**

$$\text{(IVP)} \qquad x'(t) = f(t, x(t)),\ t \in \mathbf{R}, \quad x(0) = 0.$$

(So an IVP is an ODE together with an IC.)

Suppose $0 < a < \infty$ and let

$$x_a(t) = \begin{cases} 0 & \text{if } t \le a, \\ (t-a)^p & \text{if } a < t. \end{cases}$$

Then

$$x_a'(t) = \begin{cases} 0 = f(t, x_a(t)) & \text{if } t \le a, \\ p(t-a)^{p-1} = p\left((t-a)^p\right)^{\frac{p-1}{p}} = f(t, x_a(t)) & \text{if } a < t. \end{cases}$$

That is,

$$x_a'(t) = f(t, x_a(t)),\ t \in \mathbf{R} \quad \text{and} \quad x_a(0) = 0.$$

Extra credit problem. Find *all* solutions of the IVP. For complete credit you need to be rigorous. Life will be easier if you make judicious use of the uniqueness theorem.

Remark. Example 6 on page 24 in the book is a nice example of a singular differential equation. It is a *bad* example of nonuniqueness.

Separable equations. Suppose J and K are open intervals in $\mathbf{R}$, $g :\to \mathbf{R}$ and $h : K \to \mathbf{R}$. Let

$$f(t,x) = g(t)h(x),\ (t,x) \in J \times K.$$

Consider

$$\text{(ODE)} \qquad x'(t) = f(t, x(t)) = g(t)h(x(t)).$$

(An ODE of this form is said to be **separable**.) For f to be regular it is necessary and sufficient that g be continuous and that for each $a, b \in K$ with $a < b$ there exist $L \in [0, \infty)$ such that

$$|h(x_1) - h(x_2)| \le L|x_1 - x_2| \quad \text{whenever } a \le x_1 < x_2 \le b.$$

Theorem. Suppose f is regular, I is an open subinterval of J and x is a solution of ODE on I. If $h(x(\underline{t})) = 0$ for some $\underline{t} \in I$ then

$$x(t) = x(\underline{t}) \quad \text{for all } t \in I.$$

Proof. This follows directly from the uniqueness theorem. $\square$