Test Two Mathematics 131.01 Spring 2005

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

The average was 44.47. The standard deviation was 12.13.

1. Solve

$$y^{(3)}(t) - y'(t) = e^t, \quad y(0) = y'(0) = y''(0) = 0.$$

Solution. The characteristic equation is $\lambda^3 - \lambda = 0$ the roots of which are $-1, 1, 0$ so the general solution of the homogeneous equation is

$$y_h = c_1 e^{-t} + c_2 e^t + c_3.$$

Since

$$(D^3 - D)te^t = (D+1)D(D-1)te^t = (D+1)De^t = 2e^t$$

we see that a particular solution of the original equation is

$$y_p = \frac{1}{2}te^t.$$

Letting $y = y_h + y_p$ we find that

$$\begin{aligned} y(0) &= c_1 + c_2 + c_3 + \frac{1}{2}, \\ y'(0) &= c_1 - c_2 + \frac{1}{2}, \\ y''(0) &= c_1 + c_2 + 1 \end{aligned}$$

so

$$y = -\frac{1}{4}e^{-t} - \frac{3}{4}e^t + 1 + \frac{1}{2}te^t.$$

2. Write down a first order system together with initial condition which is equivalent to the initial value problem in **1**.

Solution. Letting $y_1 = y, y_2 = y', y_3 = y''$ we find that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}'(t) = \begin{bmatrix} y_2 \\ y_3 \\ y_2 + e^t \end{bmatrix}(0), \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. Calculate y_1, y_2, y_3 in Euler's method with arbitrary step size h for

$$\frac{dy}{dx} = y^2 \quad \text{where } x_0 = 0 \text{ and } y_0 = 1.$$

Solution. Let $f(x, y) = y^2$. Then

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) = 1 + h; \\ y_2 &= y_1 + hf(x_1, y_1) = (1+h) + h(1+h)^2 = 1 + 2h + 2h^2 + h^3; \\ y_3 &= y_2 + hf(x_2, y_2) = 1 + 2h + 2h^2 + h^3 + h(1 + 2h + 2h^2 + h^3)^2 \\ &= 1 + 3h + 6h^2 + 9h^3 + 10h^4 + 8h^5 + 4h^6 + h^7. \end{aligned}$$

4. Solve

$$\ddot{y}(t) + 4y(t) = \sin 2(t-3), \quad y(0) = 0,\ \dot{y}(0) = 0.$$

(I strongly suggest you use complex exponentials.)

Solution. Let $L = D^2 + 4 = (D - 2i)(D + 2i)$.
We have

$$\sin 2(t-3) = \frac{e^{2(t-3)i} - e^{-2(t-3)i}}{2i} == ae^{2ti} + \overline{a}e^{-2ti}$$

where

$$a = \frac{e^{-6i}}{2i}.$$

Now

$$(D^2 + 4)te^{2ti} = (D + 2i)(D - 2i)te^{2ti} = (D + 2i)e^{2ti} = 4ie^{2ti}$$

and

$$(D^2 + 4)te^{-2ti} = (D - 2i)(D + 2i)te^{-2ti} = (D + 2i)e^{-2ti} = -4ie^{2ti}.$$

Thus

$$y_p = \frac{a}{4i}te^{2ti} + \frac{\overline{a}}{-4i}te^{-2ti} = -\frac{1}{4}t\cos 2(t-3)$$

is a particular solution of our equation. It follows that the general solution is

$$y = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4}t\cos 2(t-3).$$

Since

$$y(0) = c_1 \quad \text{and} \quad y'(0) = 2c_2 - \frac{1}{4}\cos(-6) = 2c_2 - \frac{1}{4}\cos(6)$$

we find that

$$y = \frac{1}{8}\cos(6)\sin 2t - \frac{1}{4}t\cos 2(t-3).$$

5. Use the method of variation of parameters to find a particular solution of

$$y''(x) + y(x) = \sec x, \quad x > 0.$$

Solution. Set $y_1 = \cos x$ and $y_2 = \sin x$. Then

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

so, by the formula on page 206,

$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{y_2(x)\sec x}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)\sec x}{W(x)}\,dx \\ &= -\cos x \int \sin x \sec x\,dx + \sin x \int \cos x \sec x\,dx \\ &= \cos x \ln \cos x + x \sin x. \end{aligned}$$

In the next two problems,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and A is the 2×2 matrix such that

$$A\mathbf{v}_1 = 4\mathbf{v}_1, \quad A\mathbf{v}_2 = -\mathbf{v}_2.$$

You do *not* need to know A to do either of these problems.

6. Determine

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

such that $\mathbf{x}(0) = \mathbf{0}$ and

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

Solution. Let

$$\mathbf{b}(t) = \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

Then

$$\mathbf{b}(t) = b_1(t)\mathbf{v}_1 + b_2(t)\mathbf{v}_2$$

where

$$b_1(t) = b_2(t) = \frac{1}{2}e^t.$$

Then

$$\begin{aligned}
\mathbf{x}(t) &= e^{tA}\left(\mathbf{x}(0) + \int_0^t e^{-\tau A}\mathbf{b}(\tau)\,d\tau\right) \\
&= \int_0^t e^{(t-\tau)A}\frac{1}{2}e^{\tau}(\mathbf{v}_1 + \mathbf{v_2})\,dt \\
&= \frac{1}{2}\left(\int_0^t e^{4(t-\tau)}e^{\tau}\,d\tau\right)\mathbf{v}_1 + \frac{1}{2}\left(\int_0^t e^{-(t-\tau)}e^{\tau}\,d\tau\right)\mathbf{v}_2 \\
&= \frac{1}{6}(e^{4t} - e^t)\mathbf{v}_1 - \frac{1}{4}(e^{-t} - e^t)\mathbf{v}_2.
\end{aligned}$$

7. Determine

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

such that

$$\ddot{\mathbf{x}} + A\mathbf{x} = \mathbf{0}$$

and

$$\mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad \text{and} \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

where x_0, y_0, x_1, y_1 are given constants. (This requires some, but not much, creativity.)

Solution. Look for solutions of the forms

$$\mathbf{x}(t) = x_1(t)\mathbf{v}_1 + x_2(t)\mathbf{v}_2$$

where x_1, x_2 are scalar function to be determined. Stuffing into the ODE we find that

$$\ddot{x_1}(t) - 4x_1(t) = 0 \quad \text{and} \quad \ddot{x_2}(t) + x_2(t) = 0$$

which implies that

$$x_1(t) = c_{11}e^{2t} + c_{12}e^{-2t} \quad \text{and} \quad x_2(t) = c_{21}\cos t + c_{22}\sin t.$$

We have

$$\mathbf{x}(0) = (c_{11} + c_{12})\mathbf{v}_1 + c_{21}\mathbf{v}_2 = \begin{bmatrix} c_{11} + c_{12} - c_{21} \\ c_{11} + c_{12} + c_{21} \end{bmatrix}$$

and

$$\dot{\mathbf{x}}(0) = (2c_{11} - 2c_{12})\mathbf{v}_1 + c_{22}\mathbf{v}_2 = \begin{bmatrix} 2c_{11} - 2c_{12} - c_{22} \\ 2c_{11} - 2c_{12} + c_{22} \end{bmatrix}$$

so

$$\begin{bmatrix} x_0 & x_1 \\ y_0 & y_1 \end{bmatrix} = \begin{bmatrix} c_{11} + c_{12} - c_{21} & 2c_{11} - 2c_{12} - c_{22} \\ c_{11} + c_{12} + c_{21} & 2c_{11} - 2c_{12} + c_{22} \end{bmatrix}$$

so

$$\begin{aligned} c_{11} &= \frac{1}{8}(2x_0 + 2y_0 + x_1 + y + 1), \\ c_{12} &= \frac{1}{8}(2x_0 + 2y_0 - x_1 - y + 1), \\ c_{21} &= \frac{1}{2}(-x_0 + y_0), \\ c_{22} &= \frac{1}{2}(-x_1 + y_1). \end{aligned}$$

8. Let $I = (-1, 1)$. Write down two independent twice continuously differentiable functions $y_1, y_2 : I \to \mathbf{R}$ such that there do *not* exist continuous functions $a_0, a_1 : I \to \mathbf{R}$ such that $Ly_i = 0$, $i = 1, 2$, where

$$Ly(x) = y''(x) + a_1(x)y'(x) + a_0(x)y(x), \quad t \in I,$$

whenever $y : I \to \mathbf{R}$ is twice continuously differentiable. (This can be done very quickly *but* you need to know what you're doing.)

Solution. Two independent solutions of a regular second order ordinary differential equation cannot meet tangentially. So all we need are independent twice continuously differentiable y_1, y_2 on I which meet tangentially. For example, $y_1(x) = 1$ and $y_2(x) = 1 + x^2$ will do. on