Test One Mathematics 131.01 Spring 2005

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

The average on the test was

$$67.84$$

and the standard deviation was

$$9.7.$$

For purposes of midterm grades, ≤ 58 was C, > 58 and < 78 was B and ≥ 78 was A.

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1. Find a solution of the IVP

$$\frac{dy}{dx} = xy^2, \quad y(0) = 1.$$

Is your solution a maximal solution?

Separating variables and integrating we obtain:

$$\frac{dy}{y^2} = x\,dx, \qquad -\frac{1}{y} = \frac{x^2}{2} + C.$$

The initial condition gives

$$-\frac{1}{1} = \frac{0^2}{2} + C$$

so $C = -1$ and

$$y = \frac{2}{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

This is the unique maximal solution of the IVP.

2. Find a solution of the IVP

$$(x^2 + 1)y' + 2xy = 2x, \quad y(0) = 1.$$

Upon dividing by $x^2 + 1$ this equation is in the form $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = \frac{2x}{x^2+1}$ and $Q(x) = \frac{2x}{x^2+1}$. We have

$$\mu(x) = \exp\left(\int_0^x P(\xi)\,d\xi\right) = \exp\left(\int_0^x \frac{2\xi}{\xi^2+1}\,d\xi\right) = \exp\left(\int_0^x d(\ln\xi^2+1)\,d\xi\right) = x^2 + 1.$$

Thus

$$y(x) = \frac{1}{\mu(x)}\left(y(0) + \int_0^x \mu(\xi)Q(\xi)\,d\xi\right) = \frac{1}{x^2+1}\left(1 + \int_0^x 2\xi\,d\xi\right) = 1.$$

Heck, I could have guessed that! Hindsight is 20-20, right?

3. Find a solution of the IVP

$$x^2y' = xy + y^2, \quad y(1) = 1.$$

Dividing by x^2 we obtain the homogeneous equation

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Substituting $y = vx$ we obtain

$$x\frac{dv}{dx} + v = v + v^2 \quad \text{or} \quad \frac{dv}{dx} = \frac{v^2}{x}$$

which after separating gives

$$\frac{dv}{v^2} = \frac{dx}{x}.$$

Integrating we obtain

$$-\frac{1}{v} = \ln x + C.$$

Since $v = y/x = 1/1 = 1$ when $x = 1$ we find that $C = -1$ so

$$v = -\frac{1}{\ln x - 1} = \frac{1}{\ln x} \quad \text{and} \quad y = vx = \frac{x}{1 - \ln x}, \quad x < e.$$

4. Find a solution of the IVP

$$y'' = 2y^3, \quad y(0) = 1, \quad y'(0) = 1.$$

Making the substitution $p = dy/dx$ we find that

$$p\frac{dp}{dy} = 2y^3$$

which after separation is

$$pdp = 2y^3\, dy.$$

Integrating we obtain

$$\frac{1}{2}p^2 = \frac{1}{2}y^4 + C.$$

Since $p = dy/dx = 1$ when $y = 1$ we find that $C = 0$ so

$$p^2 = y^4$$

so

$$\frac{dy}{dx} = p = y^2.$$

Separating we obtain

$$\frac{dy}{y^2} = dx$$

which, when integrated, gives

$$-\frac{1}{y} = x + D.$$

Since $y(0) = 1$ we find that $D = -1$ so

$$y = \frac{1}{1 - x}, \quad x < 1.$$

5. The differential equation

$$\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x)dy = 0, \quad x > 0,$$

is exact. What does this mean?

It means that with $M = x^3 + \frac{y}{x}$ and $N = y^2 + \ln x$ we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Find a function $F = F(x, y)$, $x > 0$, such that

$$\frac{\partial F}{\partial x} = x^3 + \frac{y}{x} \quad \text{and} \quad \frac{\partial F}{\partial y} = y^2 + \ln x.$$

We have

$$F(x, y) = \int M(x, y)dx = \frac{x^4}{4} + y \ln x + C(y)$$

so

$$y^2 + \ln x = N(x, y) = \frac{\partial F}{\partial y} = \ln x + C'(y)$$

so $C(y) = \frac{y^3}{3} + D$ so

$$F(x, y) = \frac{x^4}{4} + y \ln x + \frac{y^3}{3} + D.$$

There's more on the other side!

6. Let

$$f(x) = e^{-x^2/2}\left(x^3 - 4x\right), \quad -\infty < x < \infty$$

and consider

(ODE) $$x'(t) = f(x(t)).$$

(i) Identify the stable and unstable critical points of ODE.

f is negative on $(-\infty, -2)$, positive on $(-2, 0)$, negative on $(0, 2)$ and positive on $(2, \infty)$. Thus 0 is a stable critical point whereas ± 2 are unstable.

(ii) Suppose x is the unique maximal solution of ODE such that $x(0) = 1$.

(a) What is the domain of x?

It's all of **R** since $0 < x(t) < 2$ for all t by uniqueness.

(b) Do the limits as $t \to \pm\infty$ exist and, if so, what are they?

They do exist. $\lim_{t\downarrow-\infty} x(t) = 2$ and $\lim_{t\uparrow\infty} x(t) = 0$.

(c) Does there exist t in the domain of x such that $x(t) = 1/2$?

Yes, there does by the Intermediate Value Theorem because x decreases from 1 to 0 on $(0, \infty)$.

(iii) Suppose x is the unique maximal solution of ODE such that $x(0) = 3$. What is the domain of x?

The domain of x is all of **R** because $x'(t) > 0$ for t in the domain of x and

$$\int_3^\infty \frac{dx}{f(x)} = \infty$$

as $f(x)$ is asymptotic to 0 as $\lim_{x\to\infty} f(x) = 0$.

7. Find a solution of the IVP

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{|x|}y, \quad y(0) = 1.$$

Is your solution a maximal solution? What, if anything, does the existence and uniqueness theorem tell you about solutions of this IVP?

We'll answer the second question first. Since the equation is regular the existence and uniqueness theorem applies; the right hand side is continuous in both x and y and satisfies a Lipschitz condition in y; it does not satisfy a Lipschitz condition in x but that is irrelevant. Let y be the unique maximal solution of the IVP.

Since the zero function is a solution of the ODE and since $y(0) = 1 > 0$ we infer from the uniqueness theorem that y is positive on its domain. For $x > 0$ we have

$$\frac{dy}{y} = \frac{3}{2}x^{\frac{1}{2}} \tag{1}$$

and for $x < 0$ we have

$$\frac{dy}{y} = \frac{3}{2}(-x)^{\frac{1}{2}}. \tag{2}$$

Antidifferentiating (1) we find that

$$\ln y = x^{\frac{3}{2}} + C, \quad x > 0.$$

The initial condition implies $C = 0$ so

$$y = \exp(x^{\frac{3}{2}}), \quad x \geq 0.$$

Antidifferentiating (2) we find that

$$\ln y = -(-x)^{\frac{3}{2}} + C, \quad x < 0.$$

The initial condition implies $C = 0$ so

$$y = \exp(-(-x)^{\frac{3}{2}}), \quad -\infty < x < 0.$$