A nifty example. Suppose

$$A,\ B,\ C,\ D,\ E,\ F \quad \text{are positive} \tag{1}$$

and

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq 0. \tag{2}$$

Let

$$L = \{(x,y) \in \mathbf{R}^2 : Ax + By = E\} \quad \text{and} \quad M = \{(x,y) \in \mathbf{R}^2 : Cx + Dy = F\};$$

let

$$a = \frac{1}{\Delta}\begin{vmatrix} E & B \\ F & D \end{vmatrix} \quad \text{and let} \quad b = \frac{1}{\Delta}\begin{vmatrix} A & E \\ C & F \end{vmatrix};$$

and note that, by Cramer's Rule (a,b) is the unique point of intersection of L and M. Let

$$a_L = \frac{E}{A}, \quad a_M = \frac{F}{C}, \quad b_L = \frac{E}{B}, \quad b_M = \frac{F}{D};$$

note that L intersects x-axis and the y-axis at the points

$$(a_L, 0) \quad \text{and} \quad (0, b_L),$$

respectively; and that M intersects the x-axis and the y-axis at the points

$$(a_M, 0) \quad \text{and} \quad (0, b_M),$$

respectively.

Let

$$\mathcal{C} = \{(0,0), (a_L, 0), (0, b_M), (a,b)\}.$$

Suppose

$$a > 0 \quad \text{and} \quad b > 0. \tag{3}$$

Note that

$$\Delta > 0 \ \Leftrightarrow\ a_L < a_M \ \Leftrightarrow\ b_L > b_M \tag{I}$$

and

$$\Delta < 0 \ \Leftrightarrow\ a_L > a_M \ \Leftrightarrow\ b_L < b_M. \tag{II}$$

Draw a picture. Note that (II) holds if and only if $\mathcal{C}$ is the set of vertices of a convex quadrilateral.

Let

$$l(x,y) = Ax + By - E, \quad m(x,y) = Cx + Dy - F, \quad f(x,y) = -xl(x,y), \quad g(x,y) = -ym(x,y)$$

and consider

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \\ \frac{dy}{dt} &= g(x,y). \end{aligned} \tag{ODE}$$

We now proceed to give a complete qualitative analysis of (ODE).

Owing to (1) and (2) there is exactly one solution to each of the linear systems

$$\begin{aligned} x = 0, &\quad y = 0; \\ x = 0, &\quad m(x,y) = 0; \\ m(x,y) = 0, &\quad y = 0; \\ m(x,y) = 0, &\quad n(x,y) = 0 \end{aligned}$$

the set of which equals $\mathcal{C}$ and these are the critical points of (ODE). Set

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = - \begin{bmatrix} l + xl_x & xl_y \\ ym_x & m + ym_y \end{bmatrix} = - \begin{bmatrix} l + Ax & Bx \\ Cy & m + Dy \end{bmatrix}.$$

We have

$$\begin{aligned} J(0,0) &= \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}; \\ J(a_L,0) &= \begin{bmatrix} -E & * \\ 0 & C(a_M - a_L) \end{bmatrix}; \\ J(0,b_M) &= \begin{bmatrix} B(b_L - b_M) & 0 \\ * & -F \end{bmatrix}; \\ J(a,b) &= \begin{bmatrix} -Aa & -Ba \\ -Cb & -Db \end{bmatrix}; \end{aligned}$$

I use "$*$" here to denote a value which is not needed to calculate the eigenvalues. It follows that if (I) holds then

$$(0,0) \text{ is unstable}; \quad (a_L,0) \text{ is unstable}; \quad (0,b_M) \text{ is unstable}$$

and that if (II) holds

$$(0,0) \text{ is unstable}; \quad (a_L,0) \text{ is stable}; \quad (b_M,0) \text{ is stable}.$$

To investigate the stability of (a,b) we first calculate that that

$$\begin{vmatrix} -Ax - \lambda & -Bx \\ -Cy & -Dy - \lambda \end{vmatrix} = \lambda^2 + (Ax + Dy)\lambda + (AD - BC)xy$$

the roots of which are

$$\lambda_\pm = \frac{-(Ax + Dy) \pm \sqrt{(Ax + Dy)^2 - 4(AD - BD)xy}}{2}.$$

As

$$(Ax + Dy)^2 - 4(AD - BD)xy = (Ax - Dy)^2 + 4BDxy > 0$$

we find that both roots are real. If $AD - BC > 0$ we have

$$(Ax + Dy)^2 - 4(AD - BC)xy < (Ax + Dy)$$

so $\lambda_+ < 0$ and $\lambda_- < 0$ so (a,b) is stable. If $AD - BC < 0$ we have

$$(Ax + Dy)^2 - 4(AD - BD)xy > (Ax + Dy)^2$$

so $\lambda_+ > 0$ and $\lambda_- < 0$ so (a,b) is unstable.

Now we draw pictures and see that we didn't have to do a lot of this to figure out what's going on.