A nifty example. Suppose

(1) \( A, B, C, D, E, F \) are positive

and

(2) \( \Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq 0. \)

Let \( L = \{(x, y) \in \mathbb{R}^2 : Ax + By = E\} \) and \( M = \{(x, y) \in \mathbb{R}^2 : Cx + Dy = F\}; \)

let

\[
\begin{align*}
a &= \frac{1}{\Delta} \begin{vmatrix} E & B \\ F & D \end{vmatrix} \quad \text{and let} \quad b &= \frac{1}{\Delta} \begin{vmatrix} A & E \\ C & F \end{vmatrix};
\end{align*}
\]

and note that, by Cramer’s Rule \((a, b)\) is the unique point of intersection of \( L \) and \( M \). Let

\[
\begin{align*}
a_L &= \frac{E}{A}, & a_M &= \frac{F}{C}, & b_L &= \frac{E}{B}, & b_M &= \frac{F}{D};
\end{align*}
\]

note that \( L \) intersects \( x \)-axis and the \( y \)-axis at the points

\((a_L, 0)\) and \((0, b_L),\)

respectively; and that \( M \) intersects the \( x \)-axis and the \( y \)-axis at the points

\((a_M, 0)\) and \((0, b_M),\)

respectively.

Let

\( C = \{(0, 0), (a_L, 0), (0, b_M), (a, b)\}. \)

Suppose

(3) \( a > 0 \) and \( b > 0. \)

Note that

(1) \( \Delta > 0 \iff a_L < a_M \iff b_L > b_M \)

and

(2) \( \Delta < 0 \iff a_L > a_M \iff b_L < b_M. \)

Draw a picture. Note that (2) holds if and only if \( C \) is the set of vertices of a convex quadrilateral.

Let

\[
\begin{align*}
l(x, y) &= Ax + By - E, & m(x, y) &= Cx + Dy - F, & f(x, y) &= -x l(x, y), & g(x, y) &= -y m(x, y)
\end{align*}
\]

and consider

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y), & \frac{dy}{dt} &= g(x, y).
\end{align*}
\]
We now proceed to give a complete qualitative analysis of (ODE). Owing to (1) and (2) there is exactly one solution to each of the linear systems

\[
\begin{align*}
x &= 0, \quad y = 0; \\
x &= 0, \quad m(x, y) = 0; \\
m(x, y) &= 0, \quad y = 0; \\
m(x, y) &= 0, \quad n(x, y) = 0
\end{align*}
\]

the set of which equals \( C \) and these are the critical points of (ODE). Set

\[
J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = - \begin{bmatrix} l + xl_x & xl_y \\ ym_x & m + ym_y \end{bmatrix} = - \begin{bmatrix} l + Ax & Bx \\ Cy & m + Dy \end{bmatrix}.
\]

We have

\[
J(0, 0) = \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}; \\
J(a_L, 0) = \begin{bmatrix} -E & * \\ 0 & C(a_M - a_L) \end{bmatrix}; \\
J(0, b_M) = \begin{bmatrix} B(b_L - b_M) & 0 \\ * & -F \end{bmatrix}; \\
J(a, b) = \begin{bmatrix} -Aa & -Ba \\ -Cb & -Db \end{bmatrix}.
\]

I use “*” here to denote a value which is not needed to calculate the eigenvalues. It follows that if (I) holds then

(0, 0) is unstable; (a_L, 0) is unstable; (0, b_M) is unstable

and that if (II) holds

(0, 0) is unstable; (a_L, 0) is stable; (b_M, 0) is stable.

To investigate the stability of (a, b) we first calculate that that

\[
\begin{vmatrix} -Ax - \lambda & -Bx \\ -Cy & -Dy - \lambda \end{vmatrix} = \lambda^2 + (Ax + Dy)\lambda + (AD - BC)xy
\]

the roots of which are

\[
\lambda_{\pm} = \frac{-(Ax + Dy) \pm \sqrt{(Ax + Dy)^2 - 4(AD - BD)xy}}{2}.
\]

As

\[(Ax + Dy)^2 - 4(AD - BD)xy = (Ax - Dy)^2 + 4BDxy > 0\]

we find that both roots are real. If \( AD - BC > 0 \) we have

\[(Ax + Dy)^2 - 4(AD - BC)xy < (Ax + Dy)^2\]

so \( \lambda_+ < 0 \) and \( \lambda_- < 0 \) so (a, b) is stable. If \( AD - BC < 0 \) we have

\[(Ax + Dy)^2 - 4(AD - BD)xy > (Ax + Dy)^2\]

so \( \lambda_+ > 0 \) and \( \lambda_- < 0 \) so (a, b) is unstable.

Now we draw pictures and see that we didn’t have to do a lot of this to figure out what’s going on.