Planetary motion.

This material comes from an article by Robert Osserman in the *American Mathematical Monthly* of July 2001.

Part One. Suppose we have an inertial frame of reference in which a body of constant mass m is concentrated at the point $\mathbf{p}$ and another body of constant mass M is concentrated at the point $\mathbf{P}$. Suppose there are no other masses in our universe and that Newton's law of gravitation holds; this amounts to

$$(1) \qquad m\ddot{\mathbf{p}} = \frac{GmM}{|\mathbf{P}-\mathbf{p}|^3}(\mathbf{P}-\mathbf{p}) \quad \text{and} \quad M\ddot{\mathbf{P}} = \frac{GmM}{|\mathbf{p}-\mathbf{P}|^3}(\mathbf{p}-\mathbf{P})$$

where G is Newton's gravitational constant in appropriate units.

Let

$$\mathbf{C} = \frac{1}{m+M}(m\mathbf{p}+M\mathbf{P})$$

be the center of mass of our two body system. A simple calculation shows that

$$\ddot{\mathbf{C}} = \mathbf{0}$$

so

$$\mathbf{C} = \mathbf{a} + t\mathbf{b}$$

for some constant vectors $\mathbf{a}$ and $\mathbf{b}$. In particular, if we translate our inertial frame by $-\mathbf{C}$ we are still in an inertial frame only now the center of mass is $\mathbf{0}$ and if we were to replace $\mathbf{P}$ and $\mathbf{p}$ by $\mathbf{P}-\mathbf{C}$ and $\mathbf{p}-\mathbf{C}$, respectively, (1) would still hold.

Let

$$\mathbf{r} = \mathbf{p} - \mathbf{P}.$$

Subtracting the first equation in (1) from the second we obtain

$$(2) \qquad \ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\mathbf{r}$$

where we have set

$$r = |\mathbf{r}|.$$

In the "standard treatment" of this subject $\mathbf{P}$ is taken to be the position of the sun and is assumed to be $\mathbf{0}$ so (1) amounts to

$$m\ddot{\mathbf{p}} = -\frac{GmM}{|\mathbf{p}|^3}\mathbf{p}$$

which has the same form as (2) but with a different constant.

. As Osserman points out, and I'm sure he was not the first to notice this, this is not correct. As Osserman goes on to say, a "correct" treatment given above allows one to discover new planets! See the article for more info on this.

Part Two. We now solve (2). Suppose $\mathbf{r}$ is a twice differentiable function defined on some open interval I of real numbers taking values in $\mathbf{R}^3 \sim \{\mathbf{0}\}$.

Let

$$\mathbf{U} = \frac{1}{r}\mathbf{r}$$

so $\mathbf{U}$ always has length one. This implies

$$\dot{\mathbf{U}} \bullet \mathbf{U} = 0.$$

We have

$$\frac{d}{dt}(\mathbf{r}\times\dot{\mathbf{r}}) = \dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{0}$$

because the cross product of a vector with itself is $\mathbf{0}$. Thus there is a constant vector $\mathbf{v}$ such that

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{v}.$$

Suppose $\mathbf{v} = \mathbf{0}$. Then, as $\dot{\mathbf{r}} = \dot{r}\mathbf{U} + r\dot{\mathbf{U}}$, we find that

$$\mathbf{U} \times \dot{\mathbf{U}} = 0.$$

This forces

$$\dot{\mathbf{U}} = 0$$

so the motion is along a straight line. We have already considered this case.

So suppose $\mathbf{v} \neq \mathbf{0}$. As $\mathbf{U} \bullet \mathbf{v} = 0$ we may replace $\mathbf{R}^3$ with $\mathbf{R}^2$.

Part Three. For a vector $(u, v) \in \mathbf{R}^2$ let

$$(u, v)^\perp = (-v, u)$$

and note that $(u, v)^\perp$ is counterclockwise rotation of (u, v) through $\pi/2$ radians.

Let

$$\theta : I \to \mathbf{R}$$

a continuous function such that

$$\mathbf{U}(t) = (\cos\theta(t), \sin\theta(t)),\ \ t \in I.$$

Using the chain rule we obtain

$$\dot{\mathbf{U}} = \dot{\theta}\mathbf{U}^\perp$$

and

$$\ddot{\mathbf{U}} = -\dot{\theta}^2\mathbf{U} + \ddot{\theta}\mathbf{U}^\perp.$$

Thus

$$\dot{\mathbf{r}} = r\dot{\mathbf{U}} = \dot{r}\mathbf{U} + r\dot{\theta}\mathbf{U}^\perp$$

and

$$\begin{aligned}
\ddot{\mathbf{r}} = r\ddot{\mathbf{U}} &= \ddot{r}\mathbf{U} + 2\dot{r}\frac{d}{dt}\mathbf{U} + r\frac{d}{dt}^2\mathbf{U} \\
&= \ddot{r}\mathbf{U} + 2\dot{r}\dot{\theta}\mathbf{U}^\perp + r\left(-\dot{\theta}^2\mathbf{U} + \ddot{\theta}\mathbf{U}^\perp\right) \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{U} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{U}^\perp.
\end{aligned}$$

Thus

(3) $$\ddot{r} - r\dot{\theta}^2 = -\frac{G(m+M)}{r^2} \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

Now

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0$$

so

(4) $$r^2\dot{\theta} = C$$

where C is some constant; $C \neq 0$ since the motion is *not* along a straight line. Replacing θ by $-\theta$ if necessary we may assume

$$C > 0.$$

This gives the Kepler law that says equal areas are swept out in equal times.

Solving (4) for $\dot\theta$ and substituting in the first equation in (3) we obtain

$$\ddot r - \frac{C^2}{r^3} = -\frac{G(m+M)}{r^2}. \tag{5}$$

How do you solve this? Let

$$z = \frac{1}{r}.$$

I bet Newton thought up this extremely clever substitution. On the other hand, if you were locked in a room and told to solve (4) you might find this substitution by trial and error, although it might take you longer than it would have taken Newton. We have

$$\frac{dr}{dt} = -\frac{1}{z^2}\frac{dz}{dt} = -r^2\frac{d\theta}{dt}\frac{dz}{d\theta} = -C\frac{dz}{d\theta}. \tag{6}$$

Differentiating one more time we get

$$\begin{aligned}\frac{d^2z}{d\theta^2} &= \frac{d}{d\theta}\left(\frac{dz}{d\theta}\right)\\ &= -\frac{1}{C}\frac{d}{d\theta}\left(\frac{dr}{dt}\right)\\ &= -\frac{1}{C}\frac{dt}{d\theta}\frac{d}{dt}\left(\frac{dr}{dt}\right)\\ &= -\frac{1}{C}\frac{r^2}{C}\left(\frac{C^2}{r^3} - \frac{G(m+M)}{r^2}\right)\\ &= -z + \frac{G(m+M)}{C^2}\end{aligned}$$

or

$$\frac{d^2z}{d\theta^2} + z = \frac{G(m+M)}{C^2}$$

the nontrivial general solution of which is

$$z = A\cos(\theta-\alpha) + \frac{G(m+M)}{C^2}$$

where A, α are constants and where $A > 0$. Thus

$$r = \frac{1}{z} = \frac{1}{A\cos(\theta-\alpha) + \frac{G(m+M)}{C^2}} = \frac{L}{1 + e\cos(\theta-\alpha)} \tag{7}$$

where

$$L = \frac{1}{A} \quad \text{and} \quad e = \frac{G(m+M)}{AC^2}.$$

Thus the orbit, which is to say the trajectory of $\mathbf{r}$, is circular if $e = 0$, elliptic if $0 < e < 1$, parabolic if $e = 1$ and hyperbolic. e is called the *eccentricity.*

Finally, from (4) we have

$$dt = \frac{r^2}{C}d\theta$$

which when integratd after substituting (7) for r says that, in case $0 \le e < 1$, the square of the period is proportional to the cube of the major semiaxis L. (I'm not so sure about this, but I'm out of time.)