Problem 41 on page 55.

Suppose $0 < T \leq \infty$; m and v are continuous $[0, T)$; m is positive and nondecreasing on $(0, T)$; v is differentiable on $(0, T)$; and

$$\text{(G)} \qquad \frac{d}{dt}(mv) = gm, \quad 0 < t < T.$$

Let

$$M(t) = \int_0^t m(\tau)\, d\tau.$$

This is equivalent to saying $M' = m$ and $M(0) = 0$. Integrating (G) from 0 to $t \in (0, T)$ we obtain

$$mv = m_0 v_0 + gM$$

where we have set $m_0 = m(0)$ and $v_0 = v(0)$. Dividing by m we obtain

$$v = \frac{m_0 v_0}{m} + g\frac{M}{m}, \quad 0 < t < T.$$

Let us now assume that $m_0 = 0$; then

$$v = g\frac{M}{m}.$$

Since m is positive and nondecreasing on $(0, T)$ we have

$$0 \leq M(t) \leq t, \quad 0 < t < T$$

so

$$\lim_{t \downarrow 0} \frac{M(t)}{m(t)} = 0.$$

This forces

$$v_0 = 0.$$

We are given

$$m = \frac{4\pi}{3}\delta r^3 \text{ and } r = kt.$$

This gives

$$m = \delta\frac{4\pi}{3}r^3 = \delta\frac{4\pi}{3}(kt)^3 = \frac{4\pi}{3}\delta k^3 t^3.$$

Thus

$$v = g\frac{M}{m} = g\frac{t}{4}, \quad 0 < t < T.$$

If for some reason you don't like what I just did, here is another way. Leibniz' rule gives

$$\text{(1)} \qquad \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = \frac{4\pi}{3}\delta k^3 t^3 \frac{dv}{dt} + v\frac{4\pi}{3}\delta k^3 (3t^2)$$

and

$$\text{(2)} \qquad gm = g\frac{4\pi}{3}\delta k^3 t^3.$$

Equating (1) and (2) and dividing by $\frac{4\pi}{3}\delta k^3$ we get

$$\text{(3)} \qquad t^2\frac{dv}{dt} + 3vt^3 = gt^3.$$

We are given a solution v of (3) on the interval $I = (0, T)$ where T is the time when the hailstone hits the ground. Dividing (3) by t^3 for $t \in I$ we obtain

$$\frac{dv}{dt} + \frac{3}{t}v = g, \ t \in I. \tag{4}$$

Let t_1 be any point in I and let $v_1 = v(t_1)$. Let

$$\mu(t) = \exp\left(\int_{t_1}^{t} \frac{3}{\xi}\, d\xi\right) = \exp\left(3 \ln \frac{t}{t_1}\right) = \left(\frac{t}{t_1}\right)^3, \ t \in I.$$

Then

$$\begin{aligned} v &= \left(\frac{t_1}{t^3}\right)\left(v_1 + \int_{t_1}^{t} \left(\frac{\xi}{t_1}\right)^3 g\, d\xi\right) \\ &= \left(\frac{t_1}{t^3}\right)\left(v_1 + \frac{g}{4t_1^3}\left(t^4 - t_1^4\right)\right) \\ &= \frac{gt}{4} + \left(v_1 t_1^3 - \frac{gt_1}{4}\right)\frac{1}{t}. \end{aligned}$$

We also know that

$$\lim_{t \downarrow 0} v = 0.$$

This implies that $v_1 t_1^3 - \frac{gt_1}{4} = 0$ so

$$v = \frac{gt}{4}, \ t \in I$$

as desired.

Alternatively, and you may find this easier, start with

$$\frac{d}{dt}(mv) = gm,$$

set $m = \frac{4\pi}{3}\delta k^3 t^3$ just as we did above cancel constants and get

$$\frac{d}{dt}(t^3 v) = gt^3.$$

Integrating we have

$$t^3 v = \frac{gt^4}{4} + C$$

where C is a constant which you **must** put in here if your logic is to be correct. Dividing by t^3 we obtain

$$v = \frac{gt}{4} + \frac{C}{t^3}.$$

The only way you can have $\lim_{t\downarrow 0} v = 0$ is for C to be zero. Thus

$$v = \frac{gt}{4}.$$