A take home final exam problem.

Introduction. We have already considered the following. Let \( f \in P \) be such that
\[
    f(x) = x \quad \text{if} \ -\pi \leq x < \pi.
\]
Then \( \hat{f}(0) = 0 \) and
\[
    \hat{f}(n) = \frac{2\pi}{in} (-1)^{n+1} \quad \text{if} \ n \neq 0.
\]
(This is different from the first version in which there were mistakes!)

Note that if \( n \) is a positive integer and \( x \in \mathbb{R} \) then
\[
    \frac{1}{2\pi} \left( \frac{2\pi}{in} (-1)^{n+1} e^{inx} + \frac{2\pi}{-in} (-1)^{-n+1} e^{-inx} \right) = \frac{2(-1)^{n+1}}{n} \sin nx
\]
so that, as \( \hat{f}(0) = 0 \) we obtain
\[
    P_N f(x) = \sum_{|n| \leq N} \hat{f}(n) E_n(x) = \frac{1}{2\pi} \sum_{0 < |n| \leq N} \frac{2\pi}{in} (-1)^{n+1} e^{inx} = 2 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \sin nx.
\]

Now I would like to have a picture of how \( P_N f \) differs from \( f \), say from \(-3\pi\) to \(3\pi\) which is three periods. Here are Maple commands to this for \( N = 10 \).

This sets up displaying and plotting on my computer.

\[
> \text{plotsetup(x11)}:
> \text{with(plots)}:
\]

This calculates \( P_{10} f \) and creates an associated plot structure. ‘**’ below is exponentiation.

\[
> f:=0:
> \text{for n from 1 to 10 do: g:=f: f:=g+2*(-1)**(n+1)*sin(n*x)/n: od:}
> p:=plot(f,x=-3*Pi..3*Pi):
\]

This sets up and creates plots of \( f \) on each of the three periods \((-3\pi, -\pi), (-\pi, \pi), (\pi, 3\pi)\). I’m going to plot it over three periods so you can see what happens near the jumps. I’ll do it with dashed lines.

\[
> \text{L1:=[[-3*Pi,-Pi],[-Pi,Pi]]};
> \text{L2:=[[-Pi,-Pi],[Pi,Pi]]};
> \text{L3:=[[Pi,-Pi],[3*Pi,Pi]]};
> \text{p1:=plot(L1,linestyle=DASH)};
> \text{p2:=plot(L2,linestyle=DASH)};
> \text{p3:=plot(L3,linestyle=DASH)};
\]

Finally, the whole works is displayed together. This is when you get the picture. On my computer I can save the picture and then print it.

\[
> \text{display(p,p1,p2,p3)};
\]

Now here is what I want you to do.

I want you to do the same thing only this time \( f \in P \) is such that
\[
    f(x) = \begin{cases} 
    x + \pi & \text{if} \ -\pi \leq x < 0, \\
    \pi - x & \text{if} \ 0 \leq x < \pi.
    \end{cases}
\]
\( f \) is a so-called sawtooth function. Be careful, though. This time \( \hat{f}(0) \) is not zero and where one had sin’s above you’re going to get cos’s. To compute the Fourier coefficients you can (i) do it by hand (not recommended); (ii) use the Jump Formula (you’ll have to use it twice!); or (iii) use Maple.