Fourier series.

Preliminary material on inner products.

Suppose V is vector space over $\mathbf{C}$ and

$$(\cdot,\cdot)$$

is a **Hermitian inner product on** V. This means, by definition, that

$$(\cdot,\cdot) : V \times V \to \mathbf{C}$$

and that the following four conditions hold:

(i) $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ whenever $v_1, v_2, w \in V$;

(ii) $(cv, w) = c(v, w)$ whenever $c \in \mathbf{C}$ and $v, w \in V$;

(iii) $(w, v) = \overline{(v, w)}$ whenever $v, w \in V$;

(iv) (v, v) is a positive real number for any $v \in V \sim \{0\}$.

These conditions imply that

(v) $(v, w_1 + w_2) = (v, w_1) + (v, w_2)$ whenever $v, w_1, w_2 \in V$;

(vi) $(v, cw) = \overline{c}(v, w)$ whenever $c \in \mathbf{C}$ and $v, w \in V$;

(vii) $(0, v) = 0 = (v, 0)$ for any $v \in V$.

In view of (iv) and (vii) we may set

$$||v|| = \sqrt{(v, v)} \quad \text{for } v \in V$$

and note that

(viii) $||v|| = 0 \;\Leftrightarrow\; v = 0$.

We call $||v||$ the **norm of** v. Note that

(ix) $||cv|| = |c|||v||$ whenever $c \in \mathbf{C}$ and $v \in V$.

Suppose

$$A : V \times V \to \mathbf{R} \quad \text{and} \quad B : V \times V \to \mathbf{R}$$

are such that

(1) $$(v, w) = A(v, w) + iB(v, w) \quad \text{whenever } v, w \in V.$$

One easily verifies that

(i) A and B are bilinear over $\mathbf{R}$;

(ii) A is symmetric and positive definite;

(iii) B is antisymmetric;

(iv) $A(iv, iw) = A(v, w)$ whenever $v, w \in V$;

(v) $B(v,w) = -A(iv,w)$ whenever $v,w \in V$.

Conversely, given $A : V \times V \to \mathbf{R}$ which is bilinear over $\mathbf{R}$ and which is positive definite symmetric, letting B be as in (v) and let $(\cdot,\cdot)$ be as in (1) we find that $(\cdot,\cdot)$ is a Hermitian inner product on V. The interested reader might write down conditions on B which allow one to construct A and $(\cdot,\cdot)$ as well.

Example One. Let

$$(z,w) = \sum_{j=1}^{n} z_j \overline{w_j} \quad \text{for } z,w \in \mathbf{C}^n.$$

The $(\cdot,\cdot)$ is easily seen to be a Hermitian inner product, called the **standard (Hermitian) inner product**, on $\mathbf{C}^n$.

Example Two. Suppose $-\infty < a < b < \infty$ and $\mathcal{F}$ is the vector space of complex valued Riemann integrable functions on $[a,b]$. Note that

$$\int_a^b f(x)\,dx = \int_a^b \Re f(x)\,dx + i\int_a^b \Im f(x)\,dx.$$

Let

$$(f,g) = \int_a^b f(x)\overline{g(x)}\,dx \quad \text{whenever } f,g \in \mathcal{F}.$$

One easily verifies that (i)-(iii) of the properties of an inner product hold and that (iv) *almost* holds in the sense that for any $f \in \mathcal{F}$ we have

$$(f,f) = \int_a^b |f(x)|^2\,dx \geq 0$$

with equality only if $\{x \in [a,b] : f(x) = 0\}$ has zero Jordan content. In particular, if f is continuous and $(f,f) = 0$ then $f(x) = 0$ for all $x \in [a,b]$.

This Example is like Example One in that one can think of $f \in \mathcal{F}$ as a an infinite-tuple with the continous index $x \in [a,b]$.

The following simple Proposition is indispensable.

Proposition. Suppose $v,w \in V$. Then

$$||v+w||^2 = ||v||^2 + 2\Re(v,w) + ||w||^2.$$

Proof. We have

$$\begin{aligned} ||v+w||^2 &= (v+w,v+w) \\ &= (v,v) + (v,w) + (w,v) + (w,w) \\ &= (v,v) + (v,w) + \overline{(v,w)} + (w,w) \\ &= ||v||^2 + 2\Re(v,w) + ||w||^2. \end{aligned}$$

□

The Cauchy-Schwartz Inequality. Suppose $v,w \in \mathcal{P}$. Then

$$|(v,w)| \leq ||v||||w||$$

with eqality only if $\{v,w\}$ is dependent.

Proof. If $w = 0$ the assertion holds trivially so let us suppose $w \neq 0$. For any $c \in \mathbf{C}$ we have

$$0 \leq ||v+cw||^2 = ||v||^2 + 2\Re(v,cw) + ||cw||^2 = ||v||^2 + 2\Re(\bar{c}(v,w)) + |c|^2||w||^2.$$

Letting

$$c = -\frac{(v,w)}{||w||^2}$$

we find that

$$0 \leq ||v||^2 - \frac{|(v,w)|^2}{||w||^2}$$

with equality only if $||v + cw|| = 0$ in which case $v + cw = 0$ so $v = -cw$. $\square$

Corollary. Suppose a and b are sequences of complex numbers. Then

$$\sum_{n=0}^{\infty} |a_n b_n| \leq \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |b_n|^2 \right)^{1/2}.$$

Proof. For any nonnegative integer N apply the Cauchy-Schwartz inequality with $(\cdot, \cdot)$ equal the standard inner product on $\mathbf{C}^N$,

$$v = (a_0, \ldots, a_N) \quad \text{and} \quad w = (b_0, \ldots, b_N)$$

and then let $N \to \infty$. $\square$

The Triangle Inequality. Suppose $v, w \in \mathcal{P}$. Then

$$||v + w|| \leq ||v|| + ||w||$$

with equality only if either v is a nonnegative multiple of w or w is a nonnegative multiple of v.
Proof. Using the Cauchy-Schwartz Inequality we find that

$$||v + w||^2 = ||v||^2 + 2\Re(v,w) + ||w||^2 \leq ||v||^2 + 2||v||||w|| + ||w||^2 = (||v|| + ||w||)^2.$$

Suppose equality holds. In case $v = 0$ then $v = 0w$ so suppose $v \neq 0$. Since $|(v,w)| \geq \Re(v,w) = ||v||||w||$ we infer from the Cauchy-Schwartz Inequality that $w = cv$ for some $c \in \mathbf{C}$. Thus

$$|1 + c|||v|| = ||(1 + c)v)|| = ||v + cw|| = ||v|| + ||cw|| = (1 + |c|)||v||$$

from which we infer that

$$1 + 2\Re c + |c|^2 = |1 + c|^2 = (1 + |c|)^2 = 1 + 2|c| + |c|^2$$

which implies that c is a nonnegative real number. $\square$

Definition. Suppose U is a linear subspace of V. We let

$$U^\perp = \{v \in V : (u,v) = 0 \text{ for all } u \in U\}$$

and note that $U^\perp$ is a linear subspace of V. It follows directly from (iv) that

$$U \cap U^\perp = \{0\}.$$

Definition. We say a subset A of V is **orthonormal** if whenever $v, w \in A$ we have

$$(v,w) = \begin{cases} 1 & \text{if } v = w; \\ 0 & \text{if } v \neq w. \end{cases}$$

Orthogonal projections.

Definition. We say a linear subspace of V is **nice** if for each $v \in V$ there is $u \in U$ such that

(2) $$||v - u|| \leq ||v - \tilde{u}|| \quad \text{whenever } \tilde{u} \in U;$$

that is, no point of U is closer to v than u.

Suppose U is a nice linear subspace of V.

Proposition. Suppose $v \in V$ and $u \in U$. Then (2) holds if and only if

$$v - u \in U^{\perp}. \tag{3}$$

Proof. We have

$$||v-w||^2 = ||(v-u)+(u-w)||^2 = ||v-u||^2 + 2\Re(v-u,w) + ||u-w||^2 \quad \text{for } w \in V. \tag{4}$$

If (3) holds and $\tilde{u} \in U$ we set $w = \tilde{u}$ in (4) and infer that

$$||v-\tilde{u}||^2 = ||v-u||^2 + ||u-\tilde{u}||^2 \geq ||v-u||^2$$

so (2) holds.

Suppose (2) holds and $\tilde{u} \in U$. For and $t \in \mathbf{R}$ and $c \in \mathbf{C}$ we set $w = u - tc\tilde{u} \in U$ in (4) and obtain

$$||v-u||^2 \leq ||v-w||^2 = ||v-u||^2 + 2t\Re(v-u,c\tilde{u}) + t^2||c\tilde{u}||^2$$

so

$$0 \leq 2t\Re(v-u,c\tilde{u}) + t^2||c\tilde{u}||^2$$

so $\Re(v-u,c\tilde{u}) = 0$. Letting $c = \pm i$ we infer that $(v-u,\tilde{u}) = 0$. □

Proposition. Suppose $v \in V$ and, for $j = 1, 2$, we have

$$u_j \in U \quad \text{and} \quad ||v-u_j|| \leq ||v-\tilde{u}|| \quad \text{whenever } \tilde{u} \in U.$$

Then

$$u_1 = u_2.$$

Proof. From the preceding Proposition we obtain

$$(v-u_1,\tilde{u}) = 0 \quad \text{and} \quad (v-u_2,\tilde{u}) = 0 \quad \text{whenever } \tilde{u} \in U.$$

Subtracting we obtain

$$(u_1-u_2,\tilde{u}) = 0 \quad \text{for all } \tilde{u} \in U.$$

Now let $\tilde{u} = u_1 - u_2$. □

In view of the preceding Proposition we may define the map

$$P : V \to U,$$

callled **orthogonal projection of** V **onto** U, by requiring that

$$v - Pv \in U^{\perp} \quad \text{whenever } v \in V.$$

We define

$$P^{\perp} : V \to U^{\perp}$$

by requiring that

$$P^{\perp}v = v - Pv \quad \text{whenever } v \in V.$$

Theorem. We have

(i) P is linear;

(ii) $P \circ P = P$.

(iii) $(Pv, w) = (v, Pw)$ whenever $v, w \in P$.

(iv) $U^\perp$ is nice and $P^\perp$ is orthogonal projection of V onto $U^\perp$.

(v) $||v||^2 = ||Pv||^2 + ||P^\perp v||^2||$ for any $v \in V$.

Proof. All this follows from (3). Suppose $v, w \in V$ and $c \in \mathbf{C}$.

Then $(v + cw) - (Pv + cPw) = (v - Pv) + c(w - Pw) \in U^\perp$ so $P(v + cw) = Pv + cPw$ so (i) holds.

That $Pu = u$ for $u \in U$ is immediate and this implies (ii).

We have

$$(Pv, w) = (Pv, Pw + P^\perp w) = (Pv, Pw) = \overline{(Pw, Pv)} = \overline{(Pw, Pv + P^\perp v)} = \overline{(Pw, v)} = (v, Pw)$$

so (iii) holds.

If $w \in U^\perp$ we have

$$(v - P^\perp v, w) = (Pv, w) = 0$$

so (iv) holds.

Finally

$$||v||^2 = ||Pv + P^\perp v||^2 = ||Pv||^2 + 2\Re(Pv, P^\perp v) + ||P^\perp v||^2 = ||Pv||^2 + ||P^\perp v||^2.$$

□

The Gram-Schmidt Process. Suppose P is orthogonal projection on the nice linear subspace U of V, $\tilde{u} \in V \sim U$, $\tilde{U} = \{u + c\tilde{u} : c \in \mathbf{C}\}$ and

$$\tilde{P}v = Pv + \frac{(v, \tilde{u})}{||\tilde{u}||^2}\tilde{u} \quad \text{whenever } v \in V.$$

Then $\tilde{U}$ is nice and $\tilde{P}$ is orthogonal projection on $\tilde{U}$.

Proof. Easy exercise for the reader. □

Remark. If $U = \{0\}$ then $P = 0$ so

$$\tilde{P}(v) = \frac{(v, \tilde{u})}{||\tilde{u}||^2}\tilde{u}$$

and $\tilde{P}$ is orthogonal projection on the line $\{c\tilde{u} : c \in \mathbf{C}\}$.

Proposition. Suppose U is a finite dimensional linear subspace of V. Then U is nice.

Moreover, if B is a finite orthonormal subset of U and the number of elements in B equals the dimension of U then

$$Pv = \sum_{u \in B} (v, u)u \quad \text{and} \quad ||Pv||^2 = \sum_{u \in B} |(v, u)|^2 \quad \text{whenever } v \in V.$$

Proof. Standard linear algebra together with the Gram-Schmidt process may be used to produce B as above and to show that B is a basis for U.

Let

$$Lv = \sum_{u \in B} (v, u)u \quad \text{for } v \in V.$$

Suppose $v \in V$ and $\tilde{u} \in B$. It is evident that $(v - Lv, \tilde{u}) = 0$ which, as B is a basis for U, implies that $v - Lv \in U^\perp$; thus $P = L$.

Finally, if $v \in V$ we have

$$\begin{aligned} ||Lv||^2 &= (\sum_{u\in B}(v,u)u, \sum_{\tilde{u}\in B}(v,\tilde{u})\tilde{u}) \\ &= \sum_{u\in B,\ \tilde{u}\in B}(v,u)\overline{(v,\tilde{u})}(u,\tilde{u}) \end{aligned}$$

$$= \sum_{u\in B}|(u,v)|^2.$$

□

The real stuff.

Definition. Given a real number P, we say a complex valued function f on $\mathbf{R}$ is P**-periodic** if

$$f(x+P) = f(x) \quad \text{for all } x \in \mathbf{R}.$$

We let

$$\mathcal{P}$$

be the vector space of complex valued 2π-periodic functions on $\mathbf{R}$ which are integrable over any bounded interval.

Definition. For $f, g \in \mathcal{P}$ we let

$$(f,g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

We have already shown that $(\cdot,\cdot)$ is, essentially, a Hermitian inner product on $\mathcal{P}$.

Definition. For each $n \in \mathbf{Z}$ we let

$$E_n(x) = e^{inx}, \quad C_n \quad \text{for } x \in \mathbf{R};$$

evidently, $E_n \in \mathcal{P}$.

Proposition. Suppose $A \in \mathbf{C} \sim \{0\}$.

$$\int_a^b e^{Ax}\,dx = \frac{e^{Ax}}{A}\Big|_a^b = \frac{e^{Ab}-e^{Aa}}{A}.$$

Proof. Since

$$\frac{d}{dx}\frac{e^{Ax}}{A} = e^{Ax}, \quad \text{for } x \in \mathbf{R}$$

the Proposition follows from the Fundamental Theorem of Calculus.

Corollary

$$(E_m, E_n) = 2\pi \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{else.} \end{cases}$$

Remark. Thus the the set $\{\frac{1}{\sqrt{2\pi}}E_n : n \in \mathbf{Z}\}$ is orthonormal with respect to $(\cdot,\cdot)$.

For each $N \in \mathbf{N}$ we let

$$\mathcal{T}_N$$

be the linear subspace of $\mathcal{P}\infty$ spanned by $\{E_n : |n| \le N\}$ and we call the members of $\mathcal{T}_N$ **trigonometric polynomials of degree** N**.**

For each $f \in \mathcal{P}$ we define

$$\hat{f} : \mathbf{Z} \to \mathbf{C},$$

the **Fourier transform of** f, by letting

$$\hat{f}(n) = (f, E_n) \quad \text{for } n \in \mathbf{Z}.$$

One of our goals is to reconstruct f *from its Fourier transform.* As a first step in this direction, for each nonnegative integer N and each $f \in \mathcal{P}$ we set

$$P_N f = \frac{1}{2\pi} \sum_{|n| \le N} \hat{f}(n) E_n.$$

From our earlier work we find that if $f \in \mathcal{P}$ then $P_N f$ *is orthogonal projection of* f *onto* $\mathcal{T}_N$.

Bessel's Inequality. For any $f \in \mathcal{P}$ we have

$$\frac{1}{2\pi} \sum_{n \in \mathbf{Z}} |\hat{f}(n)|^2 \le ||f||^2.$$

Remark. Plancherel's Theorem, which comes later, will give the opposite inequality.
Proof. This follows directly from our previous work with orthogonal projections. □

Proposition. For any $f \in \mathcal{P}$ and any $a \in \mathbf{R}$ we have

$$\int_{-\pi+a}^{\pi+a} f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx.$$

Proof. Let n be that integer such that

$$2\pi n - \pi \le a < 2\pi n + \pi.$$

Making the substitution $x = w - 2\pi n$ we have

$$\int_a^{2\pi n+\pi} f(x)\,dx = \int_{a-2\pi n}^{\pi} f(w - 2\pi n)\,dw = \int_{a-2\pi n}^{\pi} f(w)\,dw.$$

Making the substitution $x = w - 2\pi(n+1)$ we have

$$\int_{2\pi n+\pi}^{a+2\pi} f(x)\,dx = \int_{-\pi}^{a-2\pi n} f(w - 2\pi(n+1))\,dw = \int_{-\pi}^{a-2\pi n} f(w)\,dw.$$

We complete the proof by adding these equations. □

Using this result we now show that the Fourier transform behaves nicely with respect to translations. Whenever $h \in \mathbf{R}$ and $f \in \mathcal{P}$ we define

$$\tau_h f : \mathbf{R} \to \mathbf{C}$$

by setting

$$\tau_h f(x) = f(x - h) \quad \text{for } x \in \mathbf{R};$$

Evidently, $\tau_h f \in \mathcal{P}$.

Proposition. We have

(1) τ_h is linear for each $h \in \mathbf{R}$;

(2) $\tau_{h_1} \circ \tau_{h_2} = \tau_{h_1+h_2}$ whenever $h_1,\ h_2 \in \mathbf{R}$;
(3) $(\tau_h f, \tau_h g) = (f, g)$ whenever $f, g \in \mathcal{P}$ and $h \in \mathbf{R}$.

Proof. Exercise. $\Box$

Proposition. Suppose $f \in \mathcal{P}$ and $h \in \mathbf{R}$. Then

$$\widehat{\tau_h f}(n) = e^{-inh}\hat{f}(n).$$

Proof. Exercise. $\Box$

Corollary. Suppose $f \in \mathcal{P}$ and $h \in \mathbf{R}$. Then

$$P_N(\tau_h f) = e^{-inh} P_N f.$$

Proof. Just unwrap the definition of P_N. $\Box$

Definition. For $f \in \mathcal{P}$ we define

$$Af : \mathbf{R} \to \mathbf{C}$$

by setting

$$Af(x) = f(-x) \quad \text{whenever } x \in \mathbf{R};$$

Evidently, $Af \in \mathcal{P}$.

We say $f \in \mathcal{P}$ is **even** if $Af = f$ and we say f is **odd** if $Af = -f$. We set

$$f_e = \frac{1}{2}(f + Af) \quad \text{and} \quad f_o = \frac{1}{2}(f - Af).$$

Evidently,

$$f_e \text{ is even}, \quad f_o \text{ is odd} \quad \text{and} \quad f = f_e + f_o.$$

Proposition. Suppose $f \in \mathcal{P}$. Then

$$\widehat{Af}(n) = -\hat{f}(n) \quad \text{for } n \in \mathbf{Z}.$$

Proof. Straightforward exercise. $\Box$

Definition. Suppose $f, g \in \mathcal{P}$. For each $x \in \mathbf{R}$ we set

$$f * g(x) = \begin{cases} \int_{-\pi}^{\pi} f(x-y)g(y)\,dy & \text{if } \int_{-\pi}^{\pi} |f(x-y)g(y)|\,dy < \infty \\ 0 & \text{else} \end{cases}$$

and we call $f * g$ the **convolution** of f and g. It is not hard to show that

$$f * g \in \mathcal{P}.$$

Proposition. Suppose $f, g \in \mathcal{P}$. Then

$$\widehat{f * g} = \hat{f}\hat{g}.$$

Proof. Exercise. $\Box$

Definition. For each nonegative integer N we define the **Dirichlet kernel** D_N by letting

$$D_N = \frac{1}{2\pi} \sum_{|n| \le N} E_n.$$

Proposition. Let N be a nonnegative integer. Then

(i)

$$D_N(x) = \frac{1}{2\pi}\begin{cases} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} & \text{if } x \neq 0 \\ 2N+1 & \text{if } x = 0; \end{cases}$$

(ii) D_N is even,

$$\int_{-\pi}^{0} D_N(x)\,dx = \frac{1}{2} = \int_0^{\pi} D_N(x)\,dx = \frac{1}{2}.$$

(iii) $P_N f = D_N * f$ for any $f \in \mathcal{P}$.

Proof. Suppose N and $x \in \mathbf{R} \sim \{0\}$. Then

$$\begin{aligned} D_N(x) &= \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} \\ &= \frac{1}{2\pi}e^{-iNx}\frac{1-(e^{ix})^{2N+1}}{1-e^{ix}} \\ &= \frac{1}{2\pi}\frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}} \\ &= \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}, \end{aligned}$$

and it is evident that $P_N(0) = \frac{2N+1}{2\pi}$ so (i) holds.

That D_N is even follows directly from (i). We have

$$2\pi\int_{-\pi}^{\pi} D_N(x)\,dx = \sum_{n=-N}^{N}(E_n, E_0) = 1$$

which, together with the fact that D_N is even implies (ii).

To prove (3), suppose $f \in \mathcal{P}$ and $x \in \mathbf{R}$ and observe that

$$\begin{aligned} P_N f(x) &= \frac{1}{2\pi}\sum_{|n|\le N}(f, E_n)E_n \\ &= \frac{1}{2\pi}\sum_{|n|\le N}\left(\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\right)e^{inx} \\ &= \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\sum_{|n|\le N} e^{in(x-t)}\right)f(t)\,dt \\ &= D_N * f(x). \end{aligned}$$

The Riemann Lebesgue Lemma. Suppose $-\infty < a < b < \infty$ and f is integrable on (a,b). Then

$$\lim_{t\to\infty}\int_a^b f(x)\sin tx\,dx = 0.$$

Proof. It follows directly from a preceding Proposition that the assertion holds if f is a step function.

Let $\eta > 0$. Choose a step function s such that

$$\int_a^b |f-s| \le \eta.$$

Then

$$\begin{aligned} |\int_a^b f(x)\sin tx\,dx| &= |\int_a^b [f(x)-s(x)]\sin tx\,dx + \int_a^b s(x)\sin tx\,dx| \\ &\le \int_a^b |f(x)-s(x)|\,dx + |\int_a^b s(x)\sin tx\,dx| \\ &\le \eta + |\int_a^b s(x)\sin tx\,dx|; \end{aligned}$$

for any $t \in \mathbf{R}$. Let $t \to \infty$ and note that η is arbitrary to complete the proof. □

Corollary. Suppose $f \in \mathcal{P}$. Then

$$\lim_{|n|\to\infty} \hat{f}(n) = 0.$$

Lemma. Suppose $0 < |x| < \pi/2$. Then

$$|\sin x| \ge \frac{1}{\sqrt{1+\pi^2}}|x|.$$

Proof. Suppose $0 < b < \pi/2$. If $0 < |x| \le b$ then, by the Mean Value Theorem there is $\xi \in (-|x|, |x|)$ such that

$$|\sin x| = |\sin x - \sin 0 = |\cos\xi(x-0)| \ge \cos b|x|.$$

If $b < |x| < \pi$ then

$$|\sin x| \ge \sin b \ge \frac{\sin b}{\pi}|x|.$$

Now let $b = \arctan\pi$. □

Lemma. Suppose $g \in \mathcal{P}$ and

$$\int_{-\pi}^{\pi} \left|\frac{g(x)}{x}\right| dx < \infty.$$

Then

$$\lim_{N\to\infty} P_N g(0) = 0.$$

Proof. Let $h : \mathbf{R} \to \mathbf{C}$ be such that $h(x) = \frac{g(x)}{\sin x/2}$ if $x \ne 0$ and $h(0) = 0$. By virtue of the preceding Lemma,

$$|h(x)| \le \frac{2}{\sqrt{1+\pi^2}}\left|\frac{g(x)}{x}\right| \quad \text{if } 0 < |x| < \pi$$

so

$$\int_{-\pi}^{\pi} |h(x)|\,dx < \infty.$$

By the Riemann-Lebesgue Lemma,

$$P_N g(0) = \int_{-\pi}^{\pi} D_N(x)g(x)\,dx = \int_{-\pi}^{\pi} h(x)\sin\left(N+\frac{1}{2}\right)x\,dx \to 0 \quad \text{as} \quad N \to \infty.$$

□

The Fourier Inversion Formula. Suppose $f \in \mathcal{P}$, $a \in \mathbf{R}$, $L^+, L^- \in \mathbf{C}$ and

$$\int_{a-\pi}^{a} \left|\frac{f(x)-L^-}{x-a}\right| dx + \int_a^{a+\pi} \left|\frac{f(x)-L^+}{x-a}\right| dx < \infty.$$

Then

$$\frac{1}{2\pi}\lim_{N\uparrow\infty} P_N f(a) = \frac{L^- + L^+}{2}.$$

Very important remark. For example, if f is differentiable at a the hypothesis holds with $L^{\pm} = f(a)$.

Proof. Let

$$A = \frac{L^- + L^+}{2}.$$

Let $s \in \mathcal{P}$ be such that

$$s(x) = \begin{cases} L^- - A & \text{if } -\pi \le x < 0, \\ L^+ - A & \text{if } 0 \le x < \pi \end{cases}$$

and note the s is odd. Let $c \in \mathcal{P}$ be such that $c(x) = A$ for $x \in \mathbf{R}$ and let

$$g = \tau_a f - s - c.$$

We have

$$\begin{aligned} \int_{-\pi}^{\pi} \left|\frac{g(w)}{w}\right| dw &= \int_{-\pi}^{0} \left|\frac{f(w+a) - L^-}{w}\right| dw + \int_{0}^{\pi} \left|\frac{f(w+a) - L^+}{w}\right| dw \\ &= \int_{a-\pi}^{a} \left|\frac{f(x) - L^-}{x-a}\right| dx \int_{a}^{a+\pi} \left|\frac{f(x) - L^+}{x-a}\right| dx \\ &< \infty. \end{aligned}$$

By the preceding Lemma we have

$$\lim_{N\to\infty} P_N g(0) = 0.$$

Suppose N is a positive integer. Since s is odd and D_N is even we have that $D_N s$ is odd so

$$P_N s(0) = \int_{-\pi}^{\pi} D_N(x) s(x)\, dx = 0.$$

Moreover,

$$P_N c(0) = A \int_{-\pi}^{\pi} D_N(x)\, dx = A.$$

It follows that

$$\lim_{N\to\infty} P_N \tau_a f(0) = A.$$

But

$$P_N \tau_a f(0) = \sum_{|n|\le N} \widehat{\tau_a f}(n) = \sum_{|n|\le N} e^{ina} \hat{f}(n) = P_N f(a).$$

□

Proposition. Suppose m is a positive integer, $f \in \mathcal{P}$ and f is m times continuously differentiable. Then

$$\widehat{f^{(m)}}(n) = (in)^m \hat{f}(n).$$

Proof. We use integration by parts to obtain

$$\begin{aligned} \hat{f}'(n) &= \int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx \\ &= \int_{-\pi}^{\pi} e^{-inx}\, d(f(x)) \\ &= e^{-inx} f(x)|_{x=-\pi}^{x=\pi} - \int_{-\pi}^{\pi} f(x)\, d(e^{-inx}) \\ &= in \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \\ &= in\, \hat{f}(n). \end{aligned}$$

Thus the Proposition holds if $m = 1$ and follows for arbitrary m by induction. $\square$

Remark. A good way to look at this is to note that

$$f' = \lim_{h \to 0} \frac{1}{h}(\tau_{-h} f - f)$$

and then to recall that

$$\tau_{-h} f = e^{inh} f.$$

Corollary. Suppose m is a positive integer, $f \in \mathcal{P}$ and f is m times continuously differentiable, N is a nonnegative integer and $x \in \mathbf{R}$. Then

$$|f(x) - P_N f(x)| \leq \frac{1}{\sqrt{\pi}} \frac{1}{N^{\frac{2m-1}{2}}} ||f^{(m)}||.$$

Proof. Note that

$$\sum_{n \in \mathbf{Z}} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n \in \mathbf{Z}} |b_n|^2 \right)^{1/2}$$

whenever a and b are complex valued functions on $\mathbf{Z}$; that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^{2m}} \leq \int_N^{\infty} \frac{dx}{x^{2m}} = \frac{1}{N^{2m-1}};$$

and that, by Bessel's Inequality,

$$\sum_{|n|>N} |\hat{f^{(m)}}(n)|^2 \leq ||f^{(m)}||^2.$$

Suppose $x \in \mathbf{R}$. From the Fourier Inversion Formula we have that

$$f(x) - P_N f(x) = \lim_{M \to \infty} \sum_{N < n \leq M} \hat{f}(n) E_n(s).$$

Moreover, Suppose O and N are positive integers and $O < N$.

$$\begin{aligned} \left| \sum_{N<|n|<O} \hat{f}(n) E_n(x) \right| &= \left| \sum_{N<|n|<O} \frac{1}{(in)^m} \hat{f^{(m)}}(n) E_n(x) \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\sum_{N<|n|<O} \frac{1}{n^{2m}} \right)^{1/2} \left(\sum_{|n|>N} |\hat{f^{(m)}}(n)|^2 \right)^{1/2} \\ &\leq \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{N^{2m-1}}} ||f^{(m)}||. \end{aligned}$$

$\square$

Corollary. Plancherel's (or is it Parseval's?) Theorem. Suppose $f \in \mathcal{P}$ and

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty.$$

$$||f||^2 = \frac{1}{2\pi} \sum_{n \in \mathbf{Z}} |\hat{f}(n)|^2.$$

Proof. That the right hand side does not exceed the left hand side is Bessel's Inequality. To prove the opposite inequality one combines an approximation argument with the Fourier Inversion Formula. □

Theorem. Suppose $f, g \in \mathcal{P}$, M is a positive integer

$$-\pi < a_1 < a_2 < \cdots < a_M \leq \pi,$$

$$a_0 = a_M - 2\pi,$$

$f'(x) = g(x)$ for x in any of the intervals

$$(a_0, a_1), (a_1, a_2), \ldots, (a_{M-1}, a_M)$$

and the limits

$$\lim_{x \uparrow a_j} g(x), \quad \lim_{x \downarrow a_j} g(x)$$

exist for $j = 1, \ldots, M$.

Then the limits

$$\lim_{x \uparrow a_j} f(x), \quad \lim_{x \downarrow a_j} f(x)$$

exist for $j = 1, \ldots, M$ and

$$(f, E_n) = \frac{1}{in}\left((g, E_n) + \sum_{j=1}^{M} J_m \overline{E_n(a_m)}\right)$$

where

$$J_j = \lim_{x \downarrow a_j} f(x) - \lim_{x \uparrow a_j} f(x), \quad j = 1, \ldots, M.$$

Proof. For any $j = 1, \ldots, M$ we have

$$\begin{aligned}
&\int_{a_{j-1}}^{a_j} f(x) e^{-inx}\, dx \\
&\quad = \int_{a_{j-1}}^{a_j} f(x) d\left(\frac{e^{-inx}}{-in}\right) dx \\
&\quad = f(x)\left(\frac{e^{-inx}}{-in}\right)\Big|_{a_{j-1}}^{a_j} - \int_{a_{j-1}}^{a_j} f'(x)\left(\frac{e^{-inx}}{-in}\right) dx \\
&\quad = \frac{1}{in}\left(\lim_{x \downarrow a_{j-1}} f(x)\overline{E_n(a_{j-1})} - \lim_{x \uparrow a_j} f(x)\overline{E_n(a_j)} + \int_{a_{j-1}}^{a_j} f'(x) e^{-inx}\, dx\right).
\end{aligned}$$

Now sum over $j = 1, \ldots, M$. □

Example. Let $f \in \mathcal{P}$ be such that

$$f(x) = x \quad \text{if } -\pi \leq x < \pi.$$

Then $\hat{f}(0) = 0$ and

$$\hat{f}(n) = \frac{2\pi}{in}(-1)^n \quad \text{if } n \neq 0.$$

To see this let $M = 1$ and let $a_1 = \pi$. Then $J_1 = -2\pi$ so if $n \neq 0$ we have

$$(f, E_n) = \frac{1}{in}\left((g, E_n) - 2\pi\overline{E_n(\pi)}\right) = 2\pi\frac{1}{in}(-1)^{n+1}$$

where $g \in \mathcal{P}$ is such that $g(x) = 1$ for $-\pi < x < \pi$.

Corollary.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Let f be the member of $\mathcal{P}$ such that $f(x) = x$ for $x \in [-\pi, \pi)$ and apply Parseval's Theorem. □