A formula for the solution of the general regular first order linear differential equation. Consider

$$y'(x) + P(x)y(x) = Q(x), \quad y(x_0) = y_0, \tag{IVP}$$

where P and Q are continuous on some open interval J, $x_0 \in J$ and $y_0 \in \mathbf{R}$.

Let

$$\mu(x) = \exp\left(\int_{x_0}^{x} P(\xi)\, d\xi\right), \quad x \in J. \tag{IF}$$

and call the function μ the **integrating factor**. Note that μ is characterized by the conditions

$$\mu' = \mu P \quad \text{and} \quad \mu(x_0) = 1. \tag{1}$$

Now

$$(\mu y)' = \mu y' + \mu' y = \mu y' + \mu P y = \mu(y' + Py). \tag{2}$$

Letting $x \in J$ (not strictly legal) and integrating (2) from x_0 to x, recalling the (IC) $y(x_0) = y_0$ and using the fact that $\mu(x_0) = 1$ we obtain

$$\mu(x)y(x) - y_0 = \int_{x_0}^{x} \mu(\xi)Q(\xi)\, d\xi$$

which immediately implies that

$$y(x) = \frac{1}{\mu(x)}\left(y_0 + \int_{x_0}^{x} \mu(\xi)Q(\xi)\, d\xi\right). \tag{The answer}$$