**Ordinary differential equations, initial conditions and integral equations.**

We fix a positive integer $n$. The first time you read this I suggest you take $n = 1$.

Suppose $\Omega$ is a subset of $\mathbf{R} \times \mathbf{R}^n$,

$$F : \Omega \to \mathbf{R}^n$$

and $F$ is continuous. We say $x$ **is a solution** of the **ordinary differential equation**

$$\text{(ODE)} \qquad x'(t) = F(t, x(t))$$

**on** $I$ if

(i) $I$ is an open interval in $\mathbf{R}$ and $x : I \to \mathbf{R}^n$;
(ii) $x$ is differentiable at each point of $I$;
(iii) $\{(t, x(t)) : t \in I\} \subset \Omega$ and
(iv) (ODE) holds for each $t \in I$.

Let us make the following *very* useful observation. Suppose

(v) $F$ is continuous;
(vi) (i) and (iii) hold and $x$ is continuous;
(vii) $t_0 \in I$, $(t_0, x_0) \in \Omega$ and $x$ satisfies the **initial condition**

$$\text{(IC)} \qquad x(t_0) = x_0.$$

Then $x$ is a solution of (ODE) on $I$ satisfying (IC) if and only if $x$ satisfies the **integral equation**

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau \quad \text{whenever } t \in I;$$

indeed, if $x$ is a solution of (ODE) satisfying (IC) just integrate (ODE) from $t_0$ to $t$ for each $t \in I$ to obtain (IE); if, on the other hand, $x$ satisfies (IE) it is clear that it satisfies (IC) and one need only differentiate (IE) to obtain (ODE).

One may then ask if solutions of the ODE exist and, if so, how many there are. We now proceed to give very satisfactory answers to these questions *provided* $F$ is **regular** which, by definition, means that $F$ is continuous and that

(viii) for each closed and bounded subset $K$ of $\Omega$ there is a nonnegative real number $L$ such that

$$|F(t, x_1) - F(t, x_2| \leq L|x_1 - x_2| \quad \text{whenever } (t, x_1), (t, x_2) \in K.$$

Note that $F$ is regular if $F$ is continuous and $F$ is continuously differentiable with respect to its second argument.

From now on we assume that $F$ is regular.

**Proposition.** Suppose

(a) $(t_0, x_0) \in \Omega$;
(b) $0 < S < \infty$, $B = \{x \in \mathbf{R}^n : |x - x_0| \leq S\}$ and $\{t_0\} \times B \subset \Omega$.

Then there exist $R, I, M, L$ such that

(c) $0 < R < \infty$, $I = \{t \in \mathbf{R} : |t - t_0| < R\}$ and $I \times B \subset \Omega$;
(d) $0 \leq M < \infty$ and

$$|F(t, x)| \leq M \quad \text{whenever } (t, x) \in I \times B.$$

(e) $0 \le L < \infty$ and

$$|F(t,x_1) - F(t,x_2)| \le L|x_1 - x_2| \quad \text{whenever } (t,x_1),(t,x_2) \in I \times B.$$

(e) $MR \le S$ and $LR < 1$.

*Proof.* Choose $R'$ such that $0 < R' < \infty$ and such that $I' \times B \subset \Omega$ where $I' = \{t \in \mathbf{R} : |t - t_0| \le R'\}$. This is possible since $\Omega$ is open and $\{x_0\} \times B$ is a closed and bounded subset of $\Omega$.

Next let $M = \sup\{|F(t,x)| : (t,x) \in I' \times B\}$; then $0 \le M < \infty$ because $F$ is continuous and $I' \times B$ is closed and bounded. Let $L$ be such that $0 \le L < \infty$ and

$$|F(t,x_1) - F(t,x_2)| \le L|x_1 - x_2| \quad \text{whenever } (t,x_1),(t,x_2) \in I' \times B;$$

such an $L$ exists because $I' \times B$ is closed and bounded and (viii) above holds.

Finally choose $R$ such that $0 < R \le S$, $MR \le S$ and $LR < 1$ and set $I = \{t \in \mathbf{R} : |t - t_0| < R\}$. □

**The operator C.** Suppose

$$t_0, x_0, R, I, S, B, M, L$$

are as in the preceding Proposition. Let $\mathcal{C}$ be the vector space of bounded and continuous functions $x : I \to \mathbf{R}^n$. For each $x \in \mathcal{C}$ let

$$||x|| = \sup\{|x(t)| : t \in I\}.$$

Note that

(I) $0 \le ||x|| < \infty$ whenever $x \in \mathcal{C}$;
(II) if $x \in \mathcal{C}$ then $||x|| = 0$ if and only if $x(t) = 0$ for all $t \in I$;
(III) if $c \in \mathbf{R}$ and $x \in \mathcal{C}$ then $||cx|| = |c|\,||x||$;
(IV) if $x_1, x_2 \in \mathcal{C}$ then $||x_1 + x_2|| \le ||x_1|| + ||x_2||$.

Let that is, $x \in \mathcal{B}$ if and only if $x \in \mathcal{C}$ and

$$|x(t) - x_0| \le S \quad \text{whenever } t \in I.$$

We define

$$\mathbf{C} : \mathcal{B} \to \mathcal{C}$$

by letting

$$\mathbf{C}(x) = x_0 + \int_{t_0}^{t} F(\tau, x(\tau))\, d\tau \quad \text{whenever } t \in I.$$

Existence and uniqueness rest on the following two simple Propositions.

**Proposition.** We have

$$x \in \mathcal{B} \;\Rightarrow\; \mathbf{C}(x) \in \mathcal{B}.$$

*Proof.* Suppose $x \in \mathcal{C}$ and $t \in [t_0, t_0 + R)$. Then

$$|\mathbf{C}(x)(t) - x_0| \le \int_{t_0}^{t} |F(\tau, x(\tau))|\, d\tau \le M|t - t_0| \le MR \le S$$

so $\mathbf{C}(x)(t) \in B$. In a similar fashion one proves that $\mathbf{C}(x)(t) \in B$ if $t \in (t_0 - R, t_0)$. □

Let

$$\theta = LR$$

and note that

$$0 \le \theta < 1.$$

**Proposition. The Lipschitz estimate.** Suppose $x_1, x_2 \in \mathcal{B}$. Then

$$||\mathbf{C}(x_1) - \mathbf{C}(x_2)|| \le \theta ||x_1 - x_2||.$$

*Proof.* For any $t \in I$ with $t \ge t_0$ we have

$$\begin{aligned} |\mathbf{C}(x_1)(t) - \mathbf{C}(x_2)(t)| &= \left| \int_{t_0}^t F(\tau, x_1(\tau)) - F(\tau, x_2(\tau)), dt \right| \\ &= \int_{t_0}^t |F(\tau, x_1(\tau)) - F(\tau, x_2(\tau))|, dt \\ &= \int_{t_0}^t L|x_1(\tau) - x_2(\tau)|, dt \\ &\le \int_{t_0}^t L||x_1 - x_2||, dt \\ &= L|t - t_0| ||x_1 - x_2|| \\ &\le LR||x_1 - x_2|| \\ &= \theta ||x_1 - x_2||. \end{aligned}$$

□

**Uniqueness.** Suppose for each $i = 1, 2$ $I_i$ is an open interval in $\mathbf{R}$ $x_i$ is a solution of (ODE) on $I_i$.

Then *either*

$$x_1(t) \ne x_2(t) \quad \text{for all } t \in I_1 \cap I_2$$

*or*

$$x_1(t) = x_2(t) \quad \text{for all } t \in I_1 \cap I_2$$

*Proof.* Suppose $t_0 \in I_1 \cap I_2$ and $x_i$, $i = 1, 2$, take on the common value $x_0$ at $t_0$. Choose $R', I', S, B, M, L$ so that (a)-(e) hold with $R, I$ there replaced by $R', I'$, respectively. Choose $R$ such that $0 < R \le R'$, $I = \{t \in \mathbf{R} : |t - t_0| < R\} \subset I_1 \cap I_2$ and

$$\{x_i(t) : i = 1, 2 \text{ and } t \in I\} \subset B;$$

this is possible because $x_i$ is continous at $t_0$, $i = 1, 2$.

Let $\mathcal{C}, \mathcal{B}, \mathbf{C}$ be as above. Note that $x_i \in \mathcal{B}$, $i = 1, 2$. By the Lipschitz estimate we have

$$||x_1 - x_2|| = ||\mathbf{C}(x_1) - \mathbf{C}(x_2)|| \le \theta ||x_1 - x_2||.$$

This implies that $||x_1 - x_2|| = 0$ so $x_1(t) = x_2(t)$ for all $t \in I$.

Thus $A = \{t \in I_1 \cap I_1 : x_1(t) = x_2(t)\}$ is open. Since $I_1 \cap I_2$, being an interval, is connected we find that *either* $A = \emptyset$ *or* $A = I_1 \cap I_2$. □

**Existence Theorem.** Suppose (a)-(e) hold and $\mathcal{C}, \mathcal{B}, \mathbf{C}$ are as above. The there is $x \in \mathcal{B}$ such that

$$\mathbf{C}(x) = x.$$

In particular, $x$ is a solution of (ODE) on $I$ satisfying (IC).

*Proof.* Let $\underline{w}$ be *any* member of $\mathcal{B}$; for example, we could set $\underline{w}(t) = x_0$ for $t \in I$. Let $w_1 = \mathbf{C}(\underline{x})$ and define $w_\nu,\ \nu = 2, 3, \ldots$, inductively by setting

$$w_{\nu+1} = \mathbf{C}(w_\nu).$$

From the Lipschitz estimate we have that

$$||w_{\nu+1} - w_\nu|| = ||\mathbf{C}(w_\nu - \mathbf{C}(w_{\nu-1})|| \le \theta ||w_\nu - w_{\nu-1}||, \quad \nu = 1, 2, \ldots$$

which in turn readily implies that

$$||w_{\nu+1} - w_\mu|| \le \theta^{\nu-\mu} ||w_1 - \underline{x}||, \text{ whenever } \nu \ge \mu = 1, 2, \ldots.$$

This implies that $w_\nu$ converges uniformly as to a member $x \in \mathcal{B}$ as $\nu \uparrow \infty$ and it is clear that

$$\mathbf{C}(x) = x$$

. □

**Example.** Let $n = 1$, let $(t_0, x_0) = (0, 1)$ and let $F(t, x) = x$ for $(t, x) \in \mathbf{R} \times \mathbf{R}$. Let $S$ be any positive real number; let $R = \frac{S}{S+1}$; let $I$ and $B$ be as in (a),(b); let

$$F(t, x) = F(t, x) = x \quad \text{for } (t, x) \in I \times B;$$

$$M = \max\{|F(t, x)| : (t, x) \in I \times B\} = S + 1;$$

note that if $L = 1$ then

$$|F(t, x_1) - F(t, x_2)| \le L|x_1 - x_2| \quad \text{whenever } (t, x_i) \in I \times S,\ i = 1, 2;$$

note that $MR = S$; and note that $\theta = LR < 1$. Let $F$ be the restriction of $F$ to $I \times B$ so $F(t, x) = x$ for $(t, x) \in I \times B$.

For each $\nu = 0, 1, 2, \ldots$ let

$$e_\nu(t) = \sum_{\mu=0}^{\nu} \frac{t^\mu}{\mu!} \quad \text{for } t \in I$$

and note that $e_\nu \in \mathcal{B}$. For any $\nu = 0, 1, 2, \ldots$ and $t \in I$ we find that

$$\begin{aligned} \mathbf{C}(e_\nu)(t) &= 1 + \int_0^t e_\nu(\tau)\, d\tau \\ &= 1 + \sum_{\mu=0}^{\nu} \int_0^t \frac{\tau^\mu}{\mu!}\, d\tau \\ &= 1 + \sum_{\mu=0}^{\nu} \frac{t^{\mu+1}}{(\mu+1)!} \\ &= e_{\nu+1}(t). \end{aligned}$$

It follows that $e_\nu$ converges uniformly on $I$ to

$$I \ni t \mapsto e^t = \sum_{\mu=0}^{\infty} \frac{t^\mu}{\mu!}$$

as $\nu \uparrow \infty$.

Maximal solutions. Suppose $(t_0, x_0) \in \Omega$ and let

$$x$$

be the union of those $w$ such that,for some open interval $J$, $w$ is a solution of (ODE) on $J$ satisfying (IC). It follows from the uniqueness statement above that $x$ is a function whose domain is an open interval $I$ and that $x$ is a solution of (ODE) on $I$. It follows from the existence assertion that $x$ is nonempty and that $x$ satisfies (IC). We call $x$ **the maximal solution of (ODE) satisfying (IC)**.

It follows from this and the uniqueness statement that $\Omega$ is the disjoint union of the ranges of the maximal solutions of (ODE).