Critical points and stability.

Suppose J is an open interval in $\mathbf{R}$ and

$$f : J \to \mathbf{R}$$

is such that whenever $a, b \in J$ and $a < b$ there is a nonnegative real number L such that

$$|f(x_1) - f(x_2)| \le L|x_1 - x_2| \quad \text{whenever } a \le x_1 \le x_2 \le b.$$

The theory we have developed guarantees the following.

Theorem. Suppose $a, b \in J$ and $a < b$. There is $\eta > 0$ such that if $a \le x_0 \le b$ and t_0 in $\mathbf{R}$ then there is a solution x of (ODE) defined on $(t_0 - \eta, t_0 + \eta)$ which satisfies $x(t_0) = x_0$.

(ODE) $$\frac{dx}{dt} = f(x).$$

Definition. We say x_0 is **critical point for ODE** if $x_0 \in J$ and

$$f(x_0) = 0.$$

We say the critical point x_0 is **stable** if for each $\epsilon > 0$ there $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset J$ and such that if $|x_1 - x_0| < \delta$ and if x is the unique maximal solution of ODE such that $x(0) = x_1$ then $[0, \infty)$ is a subset of the domain of x and

$$|x(t) - x_0| < \epsilon.$$

We say the critical point x_0 is **unstable** if x_0 is not stable.

Theorem. Suppose x_0 is a critical point for (ODE), $x_1 \in J$ and $x : I \to J$ is the unique maximal solution of (ODE) such that $x(0) = x_1$.

Then

(i) If $x_0 < x_1$ and $f(x) < 0$ on (x_0, x_1) then $[0, \infty) \subset I$ and

$$x_1 > x(t_1) > x(t_2) > x_0 \quad \text{whenever } 0 < t_1 < t_2 < \infty.$$

(ii) If $x_1 < x_0$ and $f(x) > 0$ on (x_0, x_1) then $[0, \infty) \subset I$ and

$$x_1 < x(t_1) < x(t_2) < x_0 \quad \text{whenever } 0 < t_1 < t_2 < \infty.$$

(iii) If $x_0 < x_1$ and $f(x) > 0$ on $(x_0, x_1]$ then

$$f(t) > x_1 \quad \text{whenever } t \in I \text{ and } t > 0.$$

(iv) If $x_1 < x_0$ and $f(x) < 0$ on $[x_1, x_0)$ then

$$x(t) < x_1 \quad \text{whenever } t \in I \text{ and } t > 0.$$

Proof. Extra credit exercise. Make use of the preceding Theorem. Here is a *big* hint. To handle (i), let T be the set of nonnegative t in the domain of x such that $x'(\tau) < 0$ whenever $0 \le \tau \le t$ and consider the least upper bound of T. $\square$

Remark. Note that if (i) and (ii) hold then x_0 is stable and that if either (iii) or (iv) hold x_0 is unstable.

Theorem. Suppose $t_0 \in \mathbf{R}$, $x : (t_0, \infty) \to J$ is a solution of ODE and

$$\tilde{x} = \lim_{t \uparrow \infty} x(t) \in J.$$

Then $\tilde{x}$ is a critical point for ODE.
Proof. Owing to the continuity of f we have

$$f(\tilde{x}) = \lim_{t \uparrow \infty} f(x(t)) = \lim_{t \uparrow \infty} x'(t).$$

By the mean value theorem there is for each positive integer $n > t_0$ a point $\xi_n \in (n, n+1)$ such that

$$x(n+1) - x(n) = x'(\xi_n).$$

Since the left hand member of this equation is converging to zero we find that $\lim_{n \uparrow \infty} x'(\xi_n) = 0$. Thus $f(\tilde{x}) = 0$. $\square$

Theorem. Suppose f is continously differentiable, x_0 is a critical point for ODE and

$$f'(x_0) < 0.$$

Then for each positive real number m such that

$$f'(x_0) < -m$$

there is $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset J$ and such that if $|x_1 - x_0| < \delta$ and x is the unique maximal solution of ODE such that $x(0) = x_1$ then $[0, \infty)$ is a subset of the domain of x and

$$|x(t) - x_0| \leq |x_1 - x_0| e^{-mt} \quad \text{whenever } 0 \leq t < \infty.$$

Remark. So x_0 is a stable critical point.

Proof. For each $x \in J$ set

$$q(x) = \int_0^1 f'(x_0 + t(x - x_0))\, dt.$$

Note that q is continuous and that $q(x_0 = f'(x_0)$. By the chain rule and the fundamental theorem of calculus we have

$$f(x) = f(x_0) + q(x)(x - x_0) = q(x)(x - x_0), \quad x \in J. \tag{1}$$

Owing to the continuity of q we may choose $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset J$ and such that

$$|\underline{x} - x_0| < \delta \;\Rightarrow\; q(\underline{x}) < -m.$$

Suppose $0 < |x_1 - x_0| < \delta$ and let $xI \to J$ be the unique maximal solution of ODE such that $x(0) = x_1$. By the preceding Theorem, $[0, \infty) \subset I$; $x(t) \neq x_0$ for $t \in [0, \infty)$; x is decreasing if $x_1 < x_0$; and x is increasing if $x_0 < x_1$.

Suppose $x_1 > x_0$. We have

$$\frac{d}{dt}(x(t) - x_0) = f(x(t)) < (x(t) - x_0) q(x(t)) < -(x(t) - x_0) m, \quad 0 < t < \infty.$$

Dividing by $x(t) - x_0$ and integrating from 0 to t we find that

$$\ln\left(\frac{x(t) - x_0}{x_1 - x_0}\right) < -mt$$

so that

$$x(t) - x_0 < (x_1 - x_0)e^{-mt}, \quad 0 < t < \infty.$$

Suppose $x_1 < x_0$. We have

$$\frac{d}{dt}(x_0 - x(t)) = -f(x(t)) = (x_0 - x(t))q(x(t)) < -m(x_0 - x(t)), \quad 0 < t < \infty.$$

Dividing by $x_0 - x(t)$ and integrating from 0 to t we find that

$$\ln\left(\frac{x_0 - x(t)}{x_0 - x_1}\right) < -mt$$

so that

$$x_0 - x(t) < (x_0 - x_1)e^{-mt}, \quad 0 < t < \infty.$$

□