**Time to blowup.**

Suppose $J$ is an open interval in $\mathbb{R}$ and

$$f : J \to \mathbb{R}$$

is continuously differentiable and consider

(ODE) \[ x'(t) = f(x(t)). \]

Suppose $x_0 \in \mathbb{R}$ and

(1) \[ f(x) > 0 \quad \text{whenever } x \geq x_0. \]

Let $x : I \to \mathbb{R}$ be the maximal solution of (ODE) such that $x(0) = x_0$.

**Proposition.** We have

(2) \[ x'(t) > 0 \quad \text{whenever } t \in I \text{ and } 0 \leq t. \]

**Proof.** Extra credit exercise. $\square$

Let

$$T = \sup I;$$

since $0 \in I$ by the definition of maximal solution we find that

$$0 < T \leq \infty.$$

It follows from (2) that

$$X = \lim_{t \uparrow T} x(t) \text{ exists.}$$

**Proposition.** We have

(3) \[ T < \infty \Rightarrow X = \sup J. \]

**Proof.** Extra credit exercise. $\square$

I claim that

$$X = \sup B$$

and that

(4) \[ T = \int_{x_0}^{\sup J} \frac{dx}{f(x)}. \]

We need a basic formula from calculus.

**Proposition.** Suppose $-\infty < c < d < \infty$;

$$g : [c, d] \to \mathbb{R};$$

$g$ is continuous; $-\infty < a < b < \infty$;

$$\phi : [a, b] \to [c, d];$$

$$1$$
\( \phi \) is continuous; and \( \phi \) is continuously differentiable on \((c, d)\); and

\[ \phi(a) = c \text{ and } \phi(b) = d. \]

Then

\[ \int_c^d g(y) \, dy = \int_a^b g(\phi(x))\phi'(x) \, dx. \]

**Proof.** Let \( G : [c, d] \to \mathbb{R} \) be such that \( G \) is continuous on \([c, d]\) and \( G'(y) = g(y) \) whenever \( y \in (c, d) \). (For example, we could set

\[ G(y) = \int_c^y g(\zeta) \, d\zeta, \quad y \in [c, d]. \]

Using the chain rule and the fundamental theorem of calculus we calculate

\[
\int_a^b g(\phi(x))\phi'(x) \, dx \\
= \int_a^b (G \circ \phi)'(x) \, dx \\
= G(\phi(a)) - G(\phi(b)) \\
= G(c) - G(d) \\
= \int_c^d g(y) \, dy.
\]

\( \square \)

Suppose \( 0 < t < T \). Using the preceding Proposition with \( a, b, c, d, \phi \) there equal \( 0, t, x_0, x(t), x \) we calculate

\[
t = \int_0^t 1 \, d\tau \\
(5) \\
= \int_0^t \frac{1}{f(x(\tau))} x'(\tau) \, d\tau \\
= \int_{x_0}^{x(t)} \frac{d\xi}{f(\xi)}
\]

Letting \( t \uparrow T \) in (4) we obtain

\[
(6) \\
T = \int_0^X \frac{d\xi}{f(\xi)}.
\]

Were it the case that \( X < \sup J \) we could infer from (6) that \( T < \infty \) and that would contradict (3).