The Parallel Projection Theorem.

We assume the Parallel Postulate. Let $|\cdot|$ be a length function for segments.

Theorem. Suppose L, L' are distinct lines, $o \in L$, $o' \in L'$. Let

$$P : L \to L'$$

be such that for each $p \in L$ *either* $p \in L'$ and $P(p) = p$ *or* $p \notin L'$ and $P(p)$ is that point p' in L' such that

$$\mathbf{l}(p, p') \parallel \mathbf{l}(o, o').$$

Then

$$\frac{|\mathbf{s}(P(c), P(d))|}{|\mathbf{s}(P(a), P(b))|} = \frac{|\mathbf{s}(c, d)|}{|\mathbf{s}(a, b)|} \tag{1}$$

whenever $a, b, c, d \in L$, $a \neq b$ and $c \neq d$.

Proof. Let us first suppose that $\mathbf{s}(a, b) \simeq \mathbf{s}(c, d)$. Let M be the line passing through a which is parallel to L' and let e be the point of intersection of M and $\mathbf{l}(b, P(b))$. Let N be the line passing through c which is parallel to L' and let f be the point of intersection of M and $\mathbf{l}(d, P(d))$. By ASA we find that $\mathbf{s}(a, e) \simeq \mathbf{s}(c, f)$. Because opposite sides of parallelograms are congruent we find that $\mathbf{s}(a, e) \simeq \mathbf{s}(P(a), P(b))$ and that $\mathbf{s}(c, f) \simeq \mathbf{s}(P(c), P(d))$. Thus, in this case, both sides of (1) are 1 and are therefore equal.

Next let us suppose p, q are positive integers and

$$p < \mathbf{s}(a, b) >= q < \mathbf{s}(c, d) > .$$

Let e be that point in $\mathbf{r}(a, b)$ such that $< \mathbf{s}(a, e) >= p < \mathbf{s}(a, b) >$ and let f be that point in $\mathbf{r}(c, d)$ such that $< \mathbf{s}(c, f) >= q < \mathbf{s}(c, d) >$. Note that $< \mathbf{s}(P(a), P(e)) >= p < \mathbf{s}(P(a), P(b)) >$ and $< \mathbf{s}(P(c), P(f)) >= q < \mathbf{s}(P(c), P(d)) >$. By the result obtained in the first paragraph we find that

$$\frac{|\mathbf{s}(P(c), P(d))|}{|\mathbf{s}(P(a), P(b))|} = \frac{p}{q} \frac{|\mathbf{s}(P(c), P(f))|}{|\mathbf{s}(P(a), P(e))|} = \frac{p}{q} \frac{|\mathbf{s}(c, f)|}{|\mathbf{s}(a, e)|} = \frac{|\mathbf{s}(c, d)|}{|\mathbf{s}(a, b)|}.$$

Thus (1) holds in this case.

Now suppose the ratio of $|\mathbf{s}(c, d)|$ to $|\mathbf{s}(a, b)|$ is irrational. Let $<$ be the geometric linear ordering on L such that $c < d$. Suppose $c < d_- < d < d_+$ and the ratios of $|\mathbf{s}(c, d_-)|$ and $|\mathbf{s}(c, d_+)|$ to $|\mathbf{s}(a, b)|$ are both rational. By the result of the preceding paragraph we have

$$\frac{|\mathbf{s}(c, d_-)|}{|\mathbf{s}(a, b)|} = \frac{|\mathbf{s}(P(c), P(d_-))|}{|\mathbf{s}(P(a), P(b))|} \leq \frac{|\mathbf{s}(P(c), P(d))|}{|\mathbf{s}(P(a), P(b))|} \leq \frac{|\mathbf{s}(P(c), P(d_+))|}{|\mathbf{s}(P(a), P(b))|} = \frac{|\mathbf{s}(c, d_+)|}{|\mathbf{s}(a, b)|}.$$

The result follows because we can make each of

$$\frac{|\mathbf{s}(c, d_+)|}{|\mathbf{s}(a, b)|} \quad \text{and} \quad \frac{|\mathbf{s}(c, d_-)|}{|\mathbf{s}(a, b)|}$$

as close to

$$\frac{|\mathbf{s}(c, d)|}{|\mathbf{s}(a, b)|}$$

as we wish. $\square$