Homework Two, due September 2

1. Suppose A and B are subsets of the set X. Prove the **DeMorgan Laws**:

$$X \sim (A \cup B) = (X \sim A) \cap (X \sim B) \quad \text{and} \quad X \sim (A \cap B) = (X \sim A) \cup (X \sim B).$$

1. Do Exercise 1.4 in **Sets, relations and functions.**

2. Suppose

$$f : \mathbf{R}^n \to \mathbf{R}^m.$$

We say f is **affine** if

$$f((1-t)\mathbf{x} + t\mathbf{y}) = (1-t)f(\mathbf{x}) + tf(\mathbf{y}) \quad \text{whenever } t \in \mathbf{R} \text{ and } \mathbf{x}, \mathbf{y} \in \mathbf{R}^2.$$

Show that f is affine if and only if there are an $m \times n$ matrix M and a vector $\mathbf{b}$ in $\mathbf{R}^m$ such that

$$f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}.$$

Note that in case $m = 1$ and $n = 1$ this amounts to saying there are real numbers m and b such that

$$f(x) = mx + b \quad \text{whenever } x \in \mathbf{R}.$$

BIG Hint: Suppose f is affine. Let $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$ for $\mathbf{x} \in \mathbf{R}^n$. Show that

(1) $$g(c\mathbf{x}) = cg(\mathbf{x}) \quad \text{whenever } c \in \mathbf{R} \text{ and } \mathbf{x} \in \mathbf{R}^n$$

and

(2) $$g(\mathbf{x} + \mathbf{y}) = g(\mathbf{x}) + g(\mathbf{y}) \quad \text{whenever } \mathbf{x}, \mathbf{y} \in \mathbf{R}^n.$$

Such a g is said to be **linear**. Use (1) to show (2). Let $\mathbf{b} = f(\mathbf{0})$. Let the i, j entry of M be the i-th component of $g(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the vector in $\mathbf{R}^n$ with 1 as its j-th component and zeroes for the other components; note that if $\mathbf{x} = (x_1, \ldots, x_n)$ then $\mathbf{x} = \sum_{j=1}^n x_j \mathbf{e}_j$.

3. Suppose L is a subset of $\mathbf{R}^2$. Show that L is a line in the standard model if and only if there is a nonconstant affine function f from $\mathbf{R}^2$ to $\mathbf{R}$ such that

$$L = \{\mathbf{x} \in \mathbf{R}^2 : f(\mathbf{x}) = 0\}.$$

Given a line L, describe all such affine functions.

4. Show that the betweenness axioms hold in the standard model for Euclidean geometry. Suggestion: In verifying (B3) make use of Exercise 3.