Homework Two, due September 2

**1.** Suppose $A$ and $B$ are subsets of the set $X$. Prove the **DeMorgan Laws**:

$$X \sim (A \cup B) = (X \sim A) \cap (X \sim B) \quad \text{and} \quad X \sim (A \cap B) = (X \sim A) \cup (X \sim B).$$

**1.** Do Exercise 1.4 in **Sets, relations and functions.**

**2.** Suppose

$$f : \mathbf{R}^n \to \mathbf{R}^m.$$

We say $f$ is **affine** if

$$f((1-t)\mathbf{x} + t\mathbf{y}) = (1-t)f(\mathbf{x}) + tf(\mathbf{y}) \quad \text{whenever } t \in \mathbf{R} \text{ and } \mathbf{x}, \mathbf{y} \in \mathbf{R}^2.$$

Show that $f$ is affine if and only if there are an $m \times n$ matrix $M$ and a vector $\mathbf{b}$ in $\mathbf{R}^m$ such that

$$f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}.$$

Note that in case $m = 1$ and $n = 1$ this amounts to saying there are real numbers $m$ and $b$ such that

$$f(x) = mx + b \quad \text{whenever } x \in \mathbf{R}.$$

**BIG** Hint: Suppose $f$ is affine. Let $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$ for $\mathbf{x} \in \mathbf{R}^n$. Show that

$$g(c\mathbf{x}) = cg(\mathbf{x}) \quad \text{whenever } c \in \mathbf{R} \text{ and } \mathbf{x} \in \mathbf{R}^n \tag{1}$$

and

$$g(\mathbf{x} + \mathbf{y}) = g(\mathbf{x}) + g(\mathbf{y}) \quad \text{whenever } \mathbf{x}, \mathbf{y} \in \mathbf{R}^n. \tag{2}$$

Such a $g$ is said to be **linear**. Use (1) to show (2). Let $\mathbf{b} = f(\mathbf{0})$. Let the $i, j$ entry of $M$ be the $i$-th component of $g(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the vector in $\mathbf{R}^n$ with 1 as its $j$-th component and zeroes for the other components; note that if $\mathbf{x} = (x_1, \ldots, x_n)$ then $\mathbf{x} = \sum_{j=1}^n x_j \mathbf{e}_j$.

**3.** Suppose $L$ is a subset of $\mathbf{R}^2$. Show that $L$ is a line in the standard model if and only if there is a nonconstant affine function $f$ from $\mathbf{R}^2$ to $\mathbf{R}$ such that

$$L = \{\mathbf{x} \in \mathbf{R}^2 : f(\mathbf{x}) = 0\}.$$

Given a line $L$, describe all such affine functions.

**4.** Show that the betweenness axioms hold in the standard model for Euclidean geometry. Suggestion: In verifying (B3) make use of Exercise 3.