Homework Two, due September 2

1. Suppose $A$ and $B$ are subsets of the set $X$. Prove the DeMorgan Laws:

\[ X \sim (A \cup B) = (X \sim A) \cap (X \sim B) \quad \text{and} \quad X \sim (A \cap B) = (X \sim A) \cup (X \sim B). \]

1. Do Exercise 1.4 in *Sets, relations and functions*.

2. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

We say $f$ is **affine** if

\[ f((1-t)x + ty) = (1-t)f(x) + tf(y) \quad \text{whenever } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^2. \]

Show that $f$ is affine if and only if there are an $m \times n$ matrix $M$ and a vector $b$ in $\mathbb{R}^m$ such that

\[ f(x) = Mx + b. \]

Note that in case $m = 1$ and $n = 1$ this amounts to saying there are real numbers $m$ and $b$ such that

\[ f(x) = mx + b \quad \text{whenever } x \in \mathbb{R}. \]

**BIG** Hint: Suppose $f$ is affine. Let $g(x) = f(x) - f(0)$ for $x \in \mathbb{R}^n$. Show that

(1) \[ g(cx) = cg(x) \quad \text{whenever } c \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \]

and

(2) \[ g(x + y) = g(x) + g(y) \quad \text{whenever } x, y \in \mathbb{R}^n. \]

Such a $g$ is said to be **linear**. Use (1) to show (2). Let $b = f(0)$. Let the $i,j$ entry of $M$ be the $i$-th component of $g(e_j)$ where $e_j$ is the vector in $\mathbb{R}^n$ with 1 as its $j$-th component and zeroes for the other components; note that if $x = (x_1, \ldots, x_n)$ then $x = \sum_{j=1}^n x_j e_j$.

3. Suppose $L$ is a subset of $\mathbb{R}^2$. Show that $L$ is a line in the standard model if and only if there is a nonconstant affine function $f$ from $\mathbb{R}^2$ to $\mathbb{R}$ such that

\[ L = \{x \in \mathbb{R}^2 : f(x) = 0 \}. \]

Given a line $L$, describe all such affine functions.

4. Show that the betweenness axioms hold in the standard model for Euclidean geometry. Suggestion: In verifying (B3) make use of Exercise 3.