Fractional linear transformations.

**Definition.** We let $\GL(2, \mathbb{C})$ be the set of invertible $2 \times 2$ matrices
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
with complex entries. Note that

(i) The identity matrix
\[
I = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
is in $\GL(2, \mathbb{C})$.

(ii) If $A$ and $B$ are in $\GL(2, \mathbb{C})$ then $AB \in \GL(2, \mathbb{C})$.

(iii) if $A \in \GL(2, \mathbb{C})$ then
\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]
is in $\GL(2, \mathbb{C})$.

That is, $\GL(2, \mathbb{C})$ is a group of matrices.

We let $\SL(2, \mathbb{R}), \ SL^-(2, \mathbb{R})$ be the set of $A$ in $\GL(2, \mathbb{C})$ with real entries and with determinant equal to 1, $-1$, respectively. Note that $\SL(2, \mathbb{R})$ and $\SL(2, \mathbb{R}) \cup \SL^-(2, \mathbb{R})$ are subgroups of $\GL(2, \mathbb{C})$.

Recall that whenever $z \in \mathbb{C} \sim \{0\}$ we have
\[
\frac{z}{0} = \infty.
\]

We let
\[
S = \mathbb{C} \cup \{\infty\}.
\]

For each $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\GL(2, \mathbb{C})$ we define
\[
T_A : S \rightarrow S
\]
by setting
\[
T_A(z) = \begin{cases}
\frac{az + b}{cz + d} & \text{if } z \in \mathbb{C} \text{ and } cz + b \neq 0; \\
\infty & \text{if } z \in \mathbb{C} \text{ and } cz + b = 0; \\
b \frac{d}{a} & \text{if } z = \infty
\end{cases}
\]
whenever $z \in S$. Noting that
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az + b \\ cz + d \end{bmatrix}
\]
and that
\[
T_A\left(\frac{z}{w}\right) = \frac{az + bw}{cz + dw} \quad \text{whenever } z, w \in \mathbb{C} \text{ and not both } z \text{ and } w \text{ are zero}
\]
we find that
\[ T_{AB} = T_A \circ T_B \quad \text{whenever } A, B \in \text{GL}(2, \mathbb{C}) \]

Evidently,
\[ T_1 \text{ is the identity map of } S. \]

It follows that for any \( A \in \text{GL}(2, \mathbb{C}) \) we have
\[ T_A^{-1} = T_A^{-1} \]
and that \( T_A \) is a permutation of \( S \). The mappings \( T_A, A \in \text{GL}(2, \mathbb{C}) \), are called **linear fractional transformations**: in view of the foregoing, we find that the set of linear fractional transformations is a subgroup of the group of permutations of \( S \).

**Proposition.** Suppose \( A \in \text{GL}(2, \mathbb{C}) \). Then \( T_A \) is the identity map of \( S \) if and only if \( A = eI \) for some \( e \in \mathbb{C} \sim \{0\} \).

**Proof.** Straightforward exercise which we leave to the reader. \( \square \)

**A very useful construction.** Suppose \( z_1, z_2, z_3 \) are distinct complex numbers. Then
\[ S \ni z \mapsto \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} \in S \]
is a linear fractional transformation which carries the points \( z_1, z_2, z_3 \) to 0, 1, \( \infty \), respectively.

**Theorem.** Suppose \( (z_1, z_2, z_3) \) and \( (w_1, w_2, w_3) \) are sets of ordered triples of distinct points in \( S \). Then there is one and only one linear fractional transformation which carries \( z_i \) to \( w_i \), \( i = 1, 2, 3 \).

Moreover, if \( A, B \in \text{GL}(2, \mathbb{C}) \) then \( T_A = T_B \) if and only if there is a nonzero complex number \( e \) such that \( A = eB \).

**Proof.** In case \( z_1, z_2, z_3 \in \mathbb{C} \) we have just exhibited a linear fractional transformation which carries the points \( z_1, z_2, z_3 \) to 0, 1, \( \infty \), respectively; we leave it to the reader to construct a linear fractional transformation with this property in case when one of the \( z_j \)'s is \( \infty \). Let \( \sigma \) be this linear fractional transformation. We then let \( \tau \) be a linear fractional transformation which carries the points \( w_1, w_2, w_3 \) to 0, 1, \( \infty \), respectively. Then \( \tau \circ \sigma^{-1} \) is a linear fractional transformation which carries \( z_1, z_2, z_3 \) to \( w_1, w_2, w_3 \), respectively. Thus there the desired linear fractional transformation exists.

Now suppose \( A, B \in \text{GL}(2, \mathbb{C}) \) and \( T_A = T_B \). By the group property we have
\[ T_{AB^{-1}} = T_A \circ T_{B^{-1}} = T_A \circ T_B^{-1} = \iota \]
where \( \iota \) is the identity map of \( S \). Thus there is a nonzero complex number \( e \) such that \( AB^{-1} = eI \) so \( A = eB \). Conversely, if \( A = eB \) for some nonzero complex number \( e \) it is evident that \( T_A = T_B \). \( \square \)

**The upper half plane.**

We let
\[ U = \{ z \in \mathbb{C} : \Re z > 0 \} \]
and call this set of complex numbers the **upper half plane**. The upper half plane will be the points in a model of hyperbolic geometry called the **Poincaré upper half plane model or P-model**.

We let
\[ L = \{ ti : t \in (0, \infty) \}. \]

We let
\[ H^+ = \{ z \in U : \Re z > 0 \} \quad \text{and let} \quad H^- = \{ z \in U : \Re z < 0 \}. \]

It will turn out that \( L \) will be a line in the P-model and \( H(L) = \{ H^+, H^- \} \). We let
\[ T^+ = \{ ti : t \in (1, \infty) \} \quad \text{and let} \quad T^- = \{ ti : t \in (0, \infty) \}. \]
It will turn out that \( \{ T^+, T^- \} \) will be the rays in \( L \) with origin \( i \). We let
\[
F = (H^+, T^+).
\]
We call \( F \) the standard flag.

A useful calculation. Suppose \( a, b, c, d \) are real, \( z \in \mathbb{C} \) and \( cz + d \neq 0 \). Let
\[
w = \frac{az + b}{cz + d}.
\]
Then
\[
2\Re w = \frac{az + b}{cz + d} + \frac{a \overline{z} + b}{cz + d} = \frac{(az + b)(cz + d) + (a \overline{z} + b)(cz + d)}{|cz + d|^2} = \frac{ac|z|^2 + adz + bc \overline{z} + bd + ac|z|^2 + ad \overline{z} + bcz + bd}{|cz + d|^2} = \frac{2}{|cz + d|^2} ac|z|^2 + (ad + bc) \Re z + bd
\]
and
\[
2i\Im w = \frac{az + b}{cz + d} - \frac{a \overline{z} + b}{cz + d} = \frac{(az + b)(cz + d) + (a \overline{z} + b)(cz + d)}{|cz + d|^2} = \frac{ac|z|^2 + adz + bc \overline{z} + bd - (ac|z|^2 + ad \overline{z} + bcz + bd)}{|cz + d|^2} = \frac{2i}{|cz + d|^2} (ad - bc) \Im z.
\]

Theorem. Suppose \( A \in \mathrm{GL}(2, \mathbb{C}) \). The following are equivalent.

(i) \( A = eB \) for some nonzero complex number and some \( B \in \mathrm{SL}(2, \mathbb{R}) \cup \mathrm{SL}^-(2, \mathbb{R}) \).

(ii) \( T_A[U] \in \{ U, -U \} \).

(iii) \( T_A[R \cup \{ \infty \}] = R \cup \{ \infty \} \).

(iv) There are three distinct points \( x_1, x_2, x_3 \in R \cup \{ \infty \} \) such that \( \{ T_A(x_1), T_A(x_2), T_A(x_3) \} \subset R \cup \{ \infty \} \).

Moreover,
\[
B \in \mathrm{SL}(2, \mathbb{R}) \iff T_B[U] = U
\]
and
\[
B \in \mathrm{SL}^-(2, \mathbb{R}) \iff T_B[U] = -U.
\]

Proof. Suppose (iv) holds. Let \( y_i = T_A(x_i), i = 1, 2, 3 \). We assume that \( x_i, y_i \in R, i = 1, 2, 3 \), and leave the case when there are infinities for the reader to handle. Let \( \sigma, \tau \in L \) be such that
\[
\sigma(z) = \frac{z - x_1 x_2 - x_3}{z - x_3 x_2 - x_1} \quad \text{and} \quad \tau(z) = \frac{z - y_1 y_2 - y_3}{z - y_3 y_2 - y_1} \quad \text{for } z \in S.
\]
Evidently, there are $C, D \in \text{GL}(2, \mathbb{C})$ with real entries such that $\sigma = T_C$ and $\tau = T_D$. By uniqueness and the formulae already developed we find that $T_A = T_D^{-1} \circ T_C = T_{D^{-1}C}$ so there is a nonzero complex number $f$ such that $A = fD^{-1}C$. Let $g$ be the reciprocal of the nonnegative square root of the absolute value of determinant of $D^{-1}C$ and let $B = g^{-1}D^{-1}C$. Then $B \in \text{SL}(2, \mathbb{R}) \cup \text{SL}^-(2, \mathbb{R})$ and $A = eB$ where $e = fg$. Thus (iv) implies (i).

Suppose (i) holds. Let $a, b, c, d$ be such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $z \in \mathbb{C}$ and let $w = T_B(z) = \frac{az + b}{cz + d}$. Then by (Im) we have

$$2iz\overline{w} = \frac{2i}{|cz + d|^2} \det B \overline{z}.$$ 

Thus (i) implies (ii). (Make sure you understand why!)

Suppose (ii) holds. By a straightforward continuity argument which we omit we find that (iii) holds. Thus (ii) implies (iii).

It is trivial that (iii) implies (iv). \(\square\)

**Definition.** Suppose $G$ is a group of permutations of the set $X$. Whenever $A \subset X$ we let

$$G_A = \{\tau \in G : \tau[A] = A\}.$$ 

Note that $G_A$ is a subgroup of $G$ and that

$$G_{\sigma[A]} = \sigma \circ G_A \circ \sigma^{-1} \quad \text{whenever } \sigma \in G.$$ 

Of particular interest is the case when $A$ has exactly one point. If $p \in X$ we set

$$G_p = G_{\{p\}}$$ 

and call this subgroup of $G$ the **isotropy group of** $p$. We have

$$G_{\sigma(p)} = \sigma \circ G_p \circ \sigma^{-1} \quad \text{whenever } \sigma \in G.$$ 

**Definition.** We let

$$G^+ = \{T_A|U : A \in \text{SL}(2, \mathbb{R})\}.$$ 

Note that if $A, B \in \text{SL}(2, \mathbb{R})$ then $T_A|U = T_B|U$ if and only if $A = \pm B$. We let

$$\iota$$ 

be the identity map of $U$ and we note that $\iota \in G^+$. We let

$$\rho(z) = -\overline{z} \quad \text{for } z \in U$$ 

and we note that $\rho$ is the restriction to $U$ of Euclidean reflection across $R\iota$. We let

$$G^- = \{\rho \circ \tau : \tau \in G^+\},$$ 

we note that

$$G^+ \cap G^- = \emptyset$$ 

4
and we let 
\[ G = G^+ \cup G^- . \]

A simple calculation shows that
\[ \alpha \circ \beta \in G^+ \quad \text{whenever} \quad \alpha, \beta \in G^- . \]

This readily implies that
\[ G^- = \{ \tau \circ \rho : \tau \in G^+ \} . \]

**Note that G is a group of permutations of U.**

We now proceed to show how G is the group of motions of a hyperbolic geometry on the upper half plane U. Henceforth we call a member of G a **motion**.

We set 
\[ \alpha(z) = -\frac{1}{z} \quad \text{for} \quad z \in U \]
and note that 
\[ \alpha \in G^+ . \]

For each \( \lambda \in (0, \infty) \) we let 
\[ \mu_\lambda(z) = \lambda z \quad \text{for} \quad z \in U \]
and note that 
\[ \mu_\lambda \in G^+ \]
and that 
\[ \mu_\lambda \circ \alpha = \alpha \circ \mu_\frac{1}{\lambda} . \]

We let 
\[ D = \{ \mu_\lambda : \lambda \in (0, \infty) \} \]
and note that D is an Abelian subgroup of G^+. We let 
\[ K = \{ \iota, \alpha, \rho, \rho \circ \alpha \} \]
and note that K is a four element Abelian subgroup of G the square of each element of which is \( \iota \). As we shall see, \( \alpha \) will be the half turn about \( i \); \( \rho \) will be reflection across \( \mathbf{L} \); and \( \rho \circ \alpha = \alpha \circ \rho \) will be reflection across the perpendicular bisector to \( \mathbf{L} \) which will be \( \{ z \in U : |u| = 1 \} \).

**Theorem.** Suppose \( \tau \in G \) and \( \tau \) carries two distinct members of \( \mathbf{L} \) into \( \mathbf{L} \). Then 
\[ \tau \in G_L . \]

Moreover,
\[ G_L = \{ \kappa \circ \mu_\lambda : \kappa \in K \quad \text{and} \quad \lambda \in (0, \infty) \} . \]

**Proof.** Let \( t_j \in (0, \infty), \quad j = 1, 2, \) be such that \( t_1 \neq t_2 \) and \( \tau(t_1 i) \in \mathbf{L} \).

Suppose \( \tau \in G^+ \). Then for some 
\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{R}) \]
we have \( \tau = T_A \). Let \( t \in (0, \infty) \) and let \( s = \frac{at + b}{ct + d} \). Then, by (Re), we have 
\[ 2Re = \frac{2}{|cti + d|^2} ac^2 + bd . \]
Thus

\[(1) \quad \tau(\xi) \in \mathbf{L} \iff acl^2 + bd = 0.\]

So (1) holds with \(t\) there equal to \(t_j, j = 1, 2\). One easily concludes that \(ac = 0\) and \(bd = 0\). Keeping in mind that \(ac - bd = 1\) we find that

\[\text{either } a \neq 0, \quad d = \frac{1}{a}, \quad b = c = 0 \quad \text{ or } a = d = 0, \quad b \in \{-1, 1\}, \quad c = -b.\]

Thus \(\tau[L] = \mathbf{L}\) and \(\tau = \kappa \circ \mu_\lambda\) for some \(\kappa \in \{i, \alpha\}\) and \(\lambda \in (0, \infty)\).

Suppose \(\tau \in \mathbf{G}^\circ\). Then \(\rho \circ \tau \in \mathbf{G}^+\) and \(\rho \circ \tau(t_j) = \tau(t_j) \in \mathbf{L}, j = 1, 2\). By the results of the preceding paragraph we have \(\rho \circ \tau = \kappa \circ \mu_\lambda\) for some \(\kappa \in \{i, \alpha\}\) and \(\lambda \in (0, \infty)\). Since \(\tau = \rho \circ \kappa \circ \mu_\lambda\) the proof is complete. \(\square\)

**Proposition.** Suppose \(u \in \mathbf{U}, \ |u| = 1\) and \(\theta \in (0, \pi)\) is such that \(u = e^{i\theta}\). Then

\[(*) \quad \frac{u+1}{u-1} = \cot \frac{\theta}{2} \quad \text{and} \quad \frac{u-1}{u+1} = -\tan \frac{\theta}{2}.\]

**Proof.** The existence and uniqueness of \(\theta\) is obvious. We have

\[\frac{u-1}{u+1} = i\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = i\frac{2i\sin \frac{\theta}{2}}{2\cos \frac{\theta}{2}} \]

so the second identity holds. Invert the second identity to get the first. \(\square\)

**Definition.** Suppose \(p \in \mathbf{U}\) and \(C \in \mathbf{R}\).

If \(\Re p = C\) we let

\[\mathbf{L}(p, C) = \{z \in \mathbf{U} : \Re z = C\},\]

we let

\[\mathbf{H}^+(p, C) = \{z \in \mathbf{U} : \Re z > C\}, \quad \text{we let} \quad \mathbf{H}^-(p, C) = \{z \in \mathbf{U} : \Re z < C\}\]

and we let

\[(V) \quad \tau_{p,C}(z) = \frac{z - \Re p}{3p} \quad \text{for } z \in \mathbf{U}.\]

If \(\Re p \neq C, R = |p - C|\) and \(\theta \in (0, \pi)\) is such that \(p = C + Re^{i\theta}\) we let we let

\[\mathbf{L}(p, C) = \{z \in \mathbf{U} : |z - C| = R\},\]

we let

\[\mathbf{H}^+(p, C) = \{z \in \mathbf{U} : |z - P| < R\}, \quad \text{we let} \quad \mathbf{H}^-(p, C) = \{z \in \mathbf{U} : |z - P| > R\}\]

and we let

\[\mathbf{L}(p, C) = \{z \in \mathbf{U} : |z - C| = R\},\]

and we let

\[(S) \quad \tau_{p,C}(z) = \left(-\tan \frac{\theta}{2}\right) \left(\frac{z - (C - R)}{z - (C + R)}\right) \quad \text{for } z \in \mathbf{U}.\]

The “V” stands for “vertical” and the “S” stands for “semicircular”.

**The Mapping Theorem.** Suppose \(p \in \mathbf{U}\) and \(C \in \mathbf{R}\) and \(L = \mathbf{L}(p, C)\).

If \(\Re p = C\) then \(\tau_{p,C} \in \mathbf{G}^+\) and \(\tau_{p,C}\) is the unique member of \(\mathbf{G}\) such that
(i) $\tau_{p,C}[L] = L$

(ii) $\tau_{p,C}(p) = i$

(iii) $\tau_{p,C}[\{z \in U : \Re p < \Re z\}] = H^+$.

(iv) $\tau_{p,C}[\{z \in L : \Im p < \Im z\}] = T^+$.

Moreover,

$$\tau_{p,C}(p + ti) = \frac{t}{\Im p} i \quad \text{whenever } t \in (0, \infty).$$

If $\Re p \neq C$, $R = |p - C|$, $\theta \in (0, \pi)$ is such that $p = C + \Re e^{i\theta}$ then $\tau_{p,C} \in G^+$ and $\tau_{p,C}$ is the unique member of $G$ such that

(v) $\tau_{p,C}(p) = i$;

(vi) $\tau_{p,C}[L] = L$;

(vii) $\tau_{p,C}[\{z \in U : |z - C| < R\}] = H^+$.

(viii) $\tau_{p,C}[\{z \in L : \Re p < \Re z\}] = T^+$.

Moreover,

$$\tau_{p,C}(P + Re^{i\psi}) = \frac{\tan \frac{\theta}{2}}{\tan \frac{\psi}{2}} i \quad \text{for } \psi \in (0, \pi).$$

**Proof.** Suppose $\Re p = C$. Because the matrix

$$\begin{bmatrix}
1 & -\Re p \\
\Im p & 1
\end{bmatrix}$$

has positive determinant we find that $\tau_{p,C} \in G^+$. That $\tau_{p,C}$ satisfies (i)-(iv) is evident.

Suppose $\Re p \neq C$. Let $R = |p - C|$ and let $\theta \in (0, \pi)$ be such that $p = C + \Re e^{i\theta}$. Because the matrices

$$\begin{bmatrix}
-\tan \frac{\theta}{2} & 0 \\
0 & 1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 & -(C - R) \\
1 & -(C + R)
\end{bmatrix}$$

have real entries and negative determinant we find that $\tau_{p,C} \in G^+$. Keeping in mind (*) we find that, with $u = e^{i\theta}$,

$$\tau_{p,C}(z) = i \frac{1 - u z - (C - R)}{1 + u z - (C + R)} \quad \text{for } z \in U.$$

It is evident that (v) holds.

Suppose $v \in L$. Let $\psi \in (0, \pi)$ be such that $v = C + \Re e^{i\psi}$. Keeping in mind (*) we find that

$$\tau_{p,C}(v) = \frac{\tan \frac{\theta}{2}}{\tan \frac{\psi}{2}} i$$

from which (vi),(viii) and (1) follow.

Suppose $w \in U$.

$$\frac{w - R}{w + R} + \frac{\overline{w} - R}{\overline{w} + R} = \frac{(w - R)(\overline{w} + R) + (\overline{w} - R)(w + R)}{|w + R|^2} = \frac{|w|^2 - R^2}{|w + R|^2}.$$
This implies that
\[ \tau_{p,c}(C + w) = \begin{cases} H^+ & \text{if } |w| < R; \\ H^- & \text{if } |w| > R \end{cases} \]
from which (vii) follows.

If \( \sigma \in \mathcal{G} \) satisfies (i)-(iv) or (v)-(vii). Then \( \tau_{p,c} \circ \sigma^{-1} \) carries \( \iota, L \) and \( T^+ \) to themselves so must equal \( \iota \) by our earlier work.

**Theorem.** Suppose \( L \subseteq U, \sigma \in \mathcal{G} \) is such that
\[
L = \sigma[L]
\]
and \( p \in L \). Then there is a unique \( C \in \mathcal{R} \) such that \( L = L(p, C) \).

**Proof.** Let \( q \in L \sim \{p\} \). If \( \mathfrak{R}p = \mathfrak{R}q \) let \( C = \mathfrak{R}p \); otherwise let
\[
C = \frac{|p|^2 - |q|^2}{2\mathfrak{R}p - 2\mathfrak{R}q}
\]
and note that \( |p - C| = |q - C| \).

Then \( \tau_{p,c} \circ \sigma \) carries the two distinct points \( p \) and \( q \) into \( L \) and so, by an earlier Theorem, \( \tau_{p,c}^{-1} \circ \sigma[L] = L \). Thus
\[
L(p, C) = \tau_{p,c}^{-1}[L] = \sigma[L] = L.
\]
The uniqueness of \( C \) is obvious. \( \square \)

**Theorem.** Suppose \( \sigma_i \in \mathcal{G} \) and \( L_i = \sigma_i[L], i = 1, 2, \) and \( L_1 \cap L_2 \) contains two or more points. Then \( L_1 = L_2 \).

**Definition of lines, betweenness and congruence. Verification of the axioms.** We say a subset \( L \) of \( U \) is a **Poincaré line** or **\( P \)-line** if \( L = \sigma[L] \) for some \( \sigma \in \mathcal{G} \). It follows from our preceding work that the Incidence Axioms hold and that if \( L \) is a \( P \)-line and \( p \in L \) then \( L = L(p, C) \) for a unique \( C \in \mathcal{R} \).

Let \( L \) be a \( P \)-line. Given distinct points \( p, q, r \) on \( L \) we define \( q \) to be between \( p \) and \( r \) if
\[
\exists \sigma^{-1}(p) < \exists \sigma^{-1}(q) < \exists \sigma^{-1}(r) \quad \text{for some } \sigma \in \mathcal{G} \text{ such that } L = \sigma[L].
\]
Previously we have shown that if \( \tau \in \mathcal{G} \) and \( \tau[L] = L \) then \( \tau \) preserves the natural notion of betweenness on \( L \). Thus the above definition is independent of \( \sigma \).

We have shown if \( \tau \in \mathcal{G} \) and \( \tau[L] = L \) then either
\[
\frac{\tau(si)}{\iota} < \frac{\tau(ti)}{\iota} \iff 0 < s < t < \infty
\]
or
\[
\frac{\tau(si)}{\iota} > \frac{\tau(ti)}{\iota} \iff 0 < s < t < \infty.
\]
It follows that (B1) and (B2) hold and that
\[
(1) \quad s(\tau(si), \tau(ti)) = \tau([s, t)i] \quad \text{whenever } 0 < s < t < \infty \text{ and } \tau \in \mathcal{G}.
\]
Suppose \( \tau \in \mathcal{G} \) and \( p \) and \( q \) be distinct points in \( U \). Let \( \sigma \in \mathcal{G} \) be such that \( \iota(p, q) = \sigma[L] \) and let \( s, t \in (0, \infty) \) be such that \( \sigma(s) = p \) and \( \sigma(t) = q \). If \( s < t \) we invoke (1) twice to obtain
\[
\tau[s(p, q)] = \tau[s(\sigma(s), \sigma(t))] = \tau[\sigma^{-1}([s, t)i)] = \tau \circ \sigma^{-1}([s, t)i] = s(\tau \circ \sigma^{-1}(si), \tau \circ \sigma^{-1}(ti)) = s(\tau(p), \tau(q)).
\]
If \( s > t \) we interchange \( p \) and \( q \) and obtain the same result. Thus the members of \( \mathcal{G} \) are betweenness preserving.
Suppose $p$ and $q$ are distinct points of $U \sim L$. Then $s(p,q)$ meets $L$ if and only if $\Re p$ and $\Re q$ have opposite signs. Thus (B3) holds for the line $L$ and

$$H(L) = \{H^+, H^-\}.$$ 

Because a P-line is by definition the image of $L$ under a member of $G$ and because the members of $G$ are betweenness preserving we find that (B3) holds for any P-line.

We infer from the Mapping Theorem that if $p \in U$ and $C \in R$ then

$$H(L(p, C)) = \{H^+(p, C), H^-(p, C)\}.$$ 

It follows from the Mapping Theorem and the fact that the group of $\tau \in G$ which carry $i$ to $i$ and $L$ to $L$ is $K$ that, given two flags $F_i = (H_i, R_i), i = 1, 2$, there is exactly one member $\tau$ of $G$ such that $\tau[R_1] = R_2$ and $\tau[H_1] = H_2$. 