Fractional linear transformations.

Definition. We let $\mathbf{GL}(2,\mathbf{C})$ be the set of invertible 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with complex entries. Note that

(i) The identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is in $\mathbf{GL}(2,\mathbf{C})$.

(ii) If A and B are in $\mathbf{GL}(2,\mathbf{C})$ then $AB \in \mathbf{GL}(2,\mathbf{C})$.

(iii) if $A \in \mathbf{GL}(2,\mathbf{C})$ then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

is in $\mathbf{GL}(2,\mathbf{C})$.

That is, $\mathbf{GL}(2,\mathbf{C})$ is a *group* of matrices.
We let

$$\mathbf{SL}(2,\mathbf{R}), \quad \mathbf{SL}^-(2,\mathbf{R})$$

be the set of A in $\mathbf{GL}(2,\mathbf{C})$ with real entries and with determinant equal to 1, -1, respectively. Note that $\mathbf{SL}(2,\mathbf{R})$ and $\mathbf{SL}(2,\mathbf{R}) \cup \mathbf{SL}^-(2,\mathbf{R})$ are subgroups of $\mathbf{GL}(2,\mathbf{C})$.

Recall that whenever $z \in \mathbf{C} \sim \{0\}$ we have

$$\frac{z}{0} = \infty.$$

We let

$$\mathbf{S} = \mathbf{C} \cup \{\infty\}.$$

For each

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $\mathbf{GL}(2,\mathbf{C})$ we define

$$T_A : \mathbf{S} \to \mathbf{S}$$

by setting

$$T_A(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbf{C} \text{ and } cz+b \neq 0; \\ \infty & \text{if } z \in \mathbf{C} \text{ and } cz+b = 0; \\ \frac{b}{d} & \text{if } z = \infty \end{cases}$$

whenever $z \in \mathbf{S}$. Noting that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix}$$

and that

$$T_A(\frac{z}{w}) = \frac{az+bw}{cz+dw} \quad \text{whenever } z, w \in \mathbf{C} \text{ and not both } z \text{ and } w \text{ are zero}$$

we find that

$$T_{AB} = T_A \circ T_B \quad \text{whenever } A, B \in \mathbf{GL}(2, \mathbf{C})$$

Evidently,

$$T_I \text{ is the identity map of } \mathbf{S}.$$

It follows that for any $A \in \mathbf{GL}(2, \mathbf{C})$ we have

$$T_A^{-1} = T_{A^{-1}}$$

and that T_A is a permutation of $\mathbf{S}$. The mappings T_A, $A \in \mathbf{GL}(2, \mathbf{C})$, are called **linear fractional transformations**; in view of the foregoing, we find that the set of linear fractional transformations is a subgroup of the group of permutations of $\mathbf{S}$.

Proposition. Suppose $A \in \mathbf{GL}(2, \mathbf{C})$. Then T_A is the identity map of $\mathbf{S}$ if and only if $A = eI$ for some $e \in \mathbf{C} \sim \{0\}$.
Proof. Straightforward exercise which we leave to the reader. □

A very useful construction. Suppose z_1, z_2, z_3 are distinct complex numbers. Then

$$\mathbf{S} \ni z \mapsto \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} \in \mathbf{S}$$

is a linear fractional transformation which carries the points z_1, z_2, z_3 to $0, 1, \infty$, respectively.

Theorem. Suppose (z_1, z_2, z_3) and (w_1, w_2, w_3) are sets of ordered triples of distinct points in $\mathbf{S}$. Then there is one and only one linear fractional transformation which carries z_i to w_i, $i = 1, 2, 3$.

Moreover, if $A, B \in \mathbf{GL}(2, \mathbf{C})$ then $T_A = T_B$ if and only if there is a nonzero complex number e such that $A = eB$.
Proof. In case $z_1, z_2, z_3 \in \mathbf{C}$ we have just exhibited a linear fractional transformation which carries the points z_1, z_2, z_3 to $0, 1, \infty$, respectively; we leave it to the reader to construct a linear fractional transformation with this property in case when one of the z_j's is ∞. Let σ be this linear fractional transformation. We then let τ be a linear fractional transformation which carries the points w_1, w_2, w_3 to $0, 1, \infty$, respectively. Then $\tau \circ \sigma^{-1}$ is a linear fractional transformation which carries z_1, z_2, z_3 to w_1, w_2, w_3, respectively. Thus there the desired linear fractional transformation exists.

Now suppose $A, B \in \mathbf{GL}(2, \mathbf{C})$ and $T_A = T_B$. By the group property we have

$$T_{AB^{-1}} = T_A \circ T_{B^{-1}} = T_A \circ T_B^{-1} = \iota$$

where ι is the identity map of $\mathbf{S}$. Thus there is a nonzero complex number e such that $AB^{-1} = eI$ so $A = eB$. Conversely, if $A = eB$ for some nonzero complex number e it is evident that $T_A = T_B$. □

The upper half plane.

We let

$$\mathbf{U} = \{z \in \mathbf{C} : \Im z > 0\}$$

and call this set of complex numbers the **upper half plane**. The upper half plane will be the points in a model of hyperbolic geometry called the **Poincaré upper half plane model** or **P-model**.

We let

$$\mathbf{L} = \{ti : t \in (0, \infty)\}.$$

We let

$$\mathbf{H}^+ = \{z \in \mathbf{U} : \Re z > 0\} \quad \text{and let} \quad \mathbf{H}^- = \{z \in \mathbf{U} : \Re z < 0\}.$$

It will turn out that $\mathbf{L}$ will be a line in the P-model and $\mathbf{H}(\mathbf{L}) = \{\mathbf{H}^+, \mathbf{H}^-\}$. We let

$$\mathbf{T}^+ = \{ti : t \in (1, \infty)\} \quad \text{and let} \quad \mathbf{T}^- = \{ti : t \in (0, \infty)\}.$$

It will turn out that $\{\mathbf{T}^+, \mathbf{T}^-\}$ will be the rays in $\mathbf{L}$ with origin i. We let

$$\mathbf{F} = (\mathbf{H}^+, \mathbf{T}^+).$$

We call $\mathbf{F}$ the **standard flag**.

A useful calculation. Suppose a, b, c, d are real, $z \in \mathbf{C}$ and $cz + d \neq 0$. Let

$$w = \frac{az + b}{cz + d}.$$

Then

$$\text{(Re)} \qquad \begin{aligned} 2\Re w &= \frac{az + b}{cz + d} + \frac{a\overline{z} + b}{c\overline{z} + d} \\ &= \frac{(az + b)(c\overline{z} + d) + (a\overline{z} + b)(cz + d)}{|cz + d|^2} \\ &= \frac{ac|z|^2 + adz + bc\overline{z} + bd + ac|z|^2 + ad\overline{z} + bcz + bd}{|cz + d|^2} \\ &= \frac{2}{|cz + d|^2} ac|z|^2 + (ad + bc)\Re z + bd \end{aligned}$$

and

$$\text{(Im)} \qquad \begin{aligned} 2i\Im w &= \frac{az + b}{cz + d} - \frac{a\overline{z} + b}{c\overline{z} + d} \\ &= \frac{(az + b)(c\overline{z} + d) + (a\overline{z} + b)(cz + d)}{|cz + d|^2} \\ &= \frac{ac|z|^2 + adz + bc\overline{z} + bd - (ac|z|^2 + ad\overline{z} + bcz + bd)}{|cz + d|^2} \\ &= \frac{2i}{|cz + d|^2}(ad - bc)\Im z. \end{aligned}$$

Theorem. Suppose $A \in \mathbf{GL}(2, \mathbf{C})$. The following are equivalent.

(i) $A = eB$ for some nonzero complex number and some $B \in \mathbf{SL}(2, \mathbf{R}) \cup \mathbf{SL}^-(2, \mathbf{R})$.

(ii) $T_A[\mathbf{U}] \in \{\mathbf{U}, -\mathbf{U}\}$.

(iii) $T_A[\mathbf{R} \cup \{\infty\}] = \mathbf{R} \cup \{\infty\}$.

(iv) There are three distinct points $x_1, x_2, x_3 \in \mathbf{R} \cup \{\infty\}$ such that $\{T_A(x_1), T_A(x_2), T_A(x_3)\} \subset \mathbf{R} \cup \{\infty\}$.

Moreover,

$$B \in \mathbf{SL}(2, \mathbf{R}) \;\Leftrightarrow\; T_B[\mathbf{U}] = \mathbf{U}$$

and

$$B \in \mathbf{SL}^-(2, \mathbf{R}) \;\Leftrightarrow\; T_B[\mathbf{U}] = -\mathbf{U}.$$

Proof. Suppose (iv) holds. Let $y_i = T_A(x_i)$, $i = 1, 2, 3$. We assume that $x_i, y_i \in \mathbf{R}$, $i = 1, 2, 3$, and leave the case when there are infinities for the reader to handle. Let $\sigma, \tau \in \mathbf{L}$ be such that

$$\sigma(z) = \frac{z - x_1}{z - x_3}\frac{x_2 - x_3}{x_2 - x_1} \quad \text{and} \quad \tau(z) = \frac{z - y_1}{z - y_3}\frac{y_2 - y_3}{y_2 - y_1} \quad \text{for } z \in \mathbf{S}.$$

Evidently, there are $C, D \in \mathbf{GL}(2, \mathbf{C})$ with real entries such that $\sigma = T_C$ and $\tau = T_D$. By uniqueness and the formulae already developed we find that $T_A = T_D^{-1} \circ T_C = T_{D^{-1}C}$ so there is a nonzero complex number f such that $A = fD^{-1}C$. Let g be the reciprocal of the nonegative square root of the absolute value of determinant of $D^{-1}C$ and let $B = g^{-1}D^{-1}C$. Then $B \in \mathbf{SL}(2, \mathbf{R}) \cup \mathbf{SL}^-(2, \mathbf{R})$ and $A = eB$ where $e = fg$. Thus (iv) implies (i).

Suppose (i) holds. Let a, b, c, d be such that

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let $z \in \mathbf{C}$ and let $w = T_B(z) = \frac{az+b}{cz+d}$. Then by (Im) we have

$$2i\Im w = \frac{2i}{|cz+d|^2}\mathbf{det}\, B \Im z.$$

Thus (i) implies (ii). (Make sure you understand why!)

Suppose (ii) holds. By a straightforward continuity argument which we omit we find that (iii) holds. Thus (ii) implies (iii).

It is trivial that (iii) implies (iv). $\square$

Definition. Suppose G is a group of permutations of the set X. Whenever $A \subset X$ we let

$$G_A = \{\tau \in G : \tau[A] = A\}.$$

Note that G_A is a subgroup of G and that

$$G_{\sigma[A]} = \sigma \circ G_A \circ \sigma^{-1} \quad \text{whenever } \sigma \in G.$$

Of particular interest is the case when A has exactly one point. If $p \in X$ we set

$$G_p = G_{\{p\}}$$

and call this subroup of G the **isotropy group of** p**.** We have

$$G_{\sigma(p)} = \sigma \circ G_p \circ \sigma^{-1} \quad \text{whenever } \sigma \in G.$$

Definition. We let

$$\mathbf{G}^+ = \{T_A|\mathbf{U} : A \in \mathbf{SL}(2, \mathbf{R})\}.$$

Note that if $A, B \in \mathbf{SL}(2, \mathbf{R})$ then $T_A|\mathbf{U} = T_B|\mathbf{U}$ if and only if $A = \pm B$. We let

$$\iota$$

be the identity map of $\mathbf{U}$ and we note that $\iota \in \mathbf{G}^+$. We let

$$\rho(z) = -\overline{z} \quad \text{for } z \in \mathbf{U}$$

and we note that ρ is the restriction to $\mathbf{U}$ of Euclidean reflection across $\mathbf{R}i$. We let

$$\mathbf{G}^- = \{\rho \circ \tau : \tau \in \mathbf{G}^+\},$$

we note that

$$\mathbf{G}^+ \cap \mathbf{G}^- = \emptyset$$

and we let

$$\mathbf{G} = \mathbf{G}^+ \cup \mathbf{G}^-.$$

A simple calculation shows that

$$\alpha \circ \beta \in \mathbf{G}^+ \quad \text{whenever } \alpha, \beta \in \mathbf{G}^-.$$

This readily implies that

$$\mathbf{G}^- = \{\tau \circ \rho : \tau \in \mathbf{G}^+\}.$$

Note that $\mathbf{G}$ *is a group of permutations of* $\mathbf{U}$*.*

We now proceed to show how $\mathbf{G}$ is the group of motions of a hyperbolic geometry on the upper half plane $\mathbf{U}$. Henceforth we call a member of $\mathbf{G}$ a **motion**.

We set

$$\alpha(z) = -\frac{1}{z} \quad \text{for } z \in \mathbf{U}$$

and note that

$$\alpha \in \mathbf{G}^+.$$

For each $\lambda \in (0, \infty)$ we let

$$\mu_\lambda(z) = \lambda z \quad \text{for } z \in \mathbf{U}$$

and note that

$$\mu_\lambda \in \mathbf{G}^+$$

and that

$$\mu_\lambda \circ \alpha = \alpha \circ \mu_{\frac{1}{\lambda}}.$$

We let

$$\mathbf{D} = \{\mu_\lambda : \lambda \in (0, \infty)\}$$

and note that $\mathbf{D}$ is an Abelian subgroup of $\mathbf{G}^+$. We let

$$\mathbf{K} = \{\iota, \alpha, \rho, \rho \circ \alpha\}$$

and note that $\mathbf{K}$ is a four element Abelian subgroup of $\mathbf{G}$ the square of each element of which is ι. As we shall see, α will be the half turn about i; ρ will be reflection across $\mathbf{L}$; and $\rho \circ \alpha = \alpha \circ \rho$ will be reflection across the perpendicular bisector to $\mathbf{L}$ which will be $\{z \in \mathbf{U} : |u| = 1\}$.

Theorem. Suppose $\tau \in \mathbf{G}$ and τ carries two distinct members of $\mathbf{L}$ into $\mathbf{L}$. Then

$$\tau \in \mathbf{G}_\mathbf{L}.$$

Moreover,

$$\mathbf{G}_\mathbf{L} = \{\kappa \circ \mu_\lambda : \kappa \in \mathbf{K} \text{ and } \lambda \in (0, \infty)\}.$$

Proof. Let $t_j \in (0, \infty)$, $j = 1, 2$, be such that $t_1 \neq t_2$ and $\tau(t_j i) \in \mathbf{L}$.

Suppose $\tau \in \mathbf{G}^+$. Then for some

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2, \mathbf{R})$$

we have $\tau = T_A$. Let $t \in (0, \infty)$ and let $s = \frac{ati+b}{cti+d}$. Then, by (Re), we have

$$2\Re s = \frac{2}{|cti + d|^2} act^2 + bd.$$

Thus

$$\tau(ti) \in \mathbf{L} \;\Leftrightarrow\; act^2 + bd = 0. \tag{1}$$

So (1) holds with t there equal to t_j, $j = 1,2$. One easily concludes that $ac = 0$ and $bd = 0$. Keeping in mind that $ac - bd = 1$ we find that

$$\textit{either } a \neq 0,\ d = \frac{1}{a},\ b = c = 0 \quad \text{it or } a = d = 0,\ b \in \{-1,1\},\ c = -b.$$

Thus $\tau[\mathbf{L}] = \mathbf{L}$ and $\tau = \kappa \circ \mu_\lambda$ for some $\kappa \in \{\iota, \alpha\}$ and $\lambda \in (0,\infty)$.

Suppose $\tau \in \mathbf{G}^-$. Then $\rho \circ \tau \in \mathbf{G}^+$ and $\rho \circ \tau(t_j) = \tau(t_j) \in \mathbf{L}$, $j = 1,2$. By the results of the preceding paragraph we have $\rho \circ \tau = \kappa \circ \mu_\lambda$ for some $\kappa \in \{\iota, \alpha\}$ and $\lambda \in (0,\infty)$. Since $\tau = \rho \circ \kappa \circ \mu_\lambda$ the proof is complete. □

Proposition. Suppose $u \in \mathbf{U}$, $|u| = 1$ and $\theta \in (0,\pi)$ is such that $u = e^{i\theta}$. Then

$$i\frac{u+1}{u-1} = \cot\frac{\theta}{2} \quad \text{and} \quad i\frac{u-1}{u+1} = -\tan\frac{\theta}{2}. \tag{$*$}$$

Proof. The existence and uniqueness of θ is obvious. We have

$$i\frac{u-1}{u+1} = i\frac{e^{i\theta}-1}{e^{i\theta}+1} = i\frac{e^{i\theta}-1}{e^{i\theta}+1}\frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}} = i\frac{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}} = i\frac{2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

so the second identity holds. Invert the second identity to get the first. □

Definition. Suppose $p \in \mathbf{U}$ and $C \in \mathbf{R}$.

If $\Re p = C$ we let

$$\mathbf{L}(p,C) = \{z \in \mathbf{U} : \Re z = C\},$$

we let

$$\mathbf{H}^+(p,C) = \{z \in \mathbf{U} : \Re z > C\}, \quad \text{we let} \quad \mathbf{H}^-(p,C) = \{z \in \mathbf{U} : \Re z < C\}$$

and we let

$$\tau_{p,C}(z) = \frac{z - \Re p}{\Im p} \quad \text{for } z \in \mathbf{U}. \tag{V}$$

If $\Re p \neq C$, $R = |p - C|$ and $\theta \in (0,\pi)$ is such that $p = C + Re^{i\theta}$ we let we let

$$\mathbf{L}(p,C) = \{z \in \mathbf{U} : |z - C| = R\},$$

we let

$$\mathbf{H}^+(p,C) = \{z \in \mathbf{U} : |z - P| < R\}, \quad \text{we let} \quad \mathbf{H}^-(p,C) = \{z \in \mathbf{U} : |z - P| > R\}$$

and we let

$$\mathbf{L}(p,C) = \{z \in \mathbf{U} : |z - C| = R\},$$

and we let

$$\tau_{p,C}(z) = \left(-\tan\frac{\theta}{2}\right)\left(\frac{z - (C-R)}{z - (C+R)}\right) \quad \text{for } z \in \mathbf{U}. \tag{S}$$

The "V" stands for "vertical" and the "S" stands for "semicircular".

The Mapping Theorem. Suppose $p \in \mathbf{U}$ and $C \in \mathbf{R}$ and $L = \mathbf{L}(p,C)$.

If $\Re p = C$ then $\tau_{p,C} \in \mathbf{G}^+$ and $\tau_{p,C}$ is the unique member of $\mathbf{G}$ such that

(i) $\tau_{p,C}[L] = \mathbf{L}$;

(ii) $\tau_{p,C}(p) = i$;

(iii) $\tau_{p,C}[\{z \in \mathbf{U} : \Re p < \Re z\}] = \mathbf{H}^+$.

(iv) $\tau_{p,C}[\{z \in L : \Im p < \Im z\}] = \mathbf{T}^+$;

Moreover,

$$\tau_{p,C}(p + ti) = \frac{t}{\Im p} i \quad \text{whenever } t \in (0, \infty).$$

If $\Re p \neq C$, $R = |p - C|$, $\theta \in (0, \pi)$ is such that $p = C + Re^{i\theta}$ then $\tau_{p,C} \in \mathbf{G}^+$ and $\tau_{p,C}$ is the unique member of $\mathbf{G}$ such that

(v) $\tau_{p,C}(p) = i$;

(vi) $\tau_{p,C}[L] = \mathbf{L}$;

(vii) $\tau_{p,C}[\{z \in \mathbf{U} : |z - C| < R\}] = \mathbf{H}^+$.

(viii) $\tau_{p,C}[\{z \in L : \Re p < \Re z\}] = \mathbf{T}^+$;

Moreover,

(1) $$\tau_{p,C}(P + Re^{i\psi}) = \frac{\tan \frac{\theta}{2}}{\tan \frac{\psi}{2}} i \quad \text{for } \psi \in (0, \pi).$$

Proof. Suppose $\Re p = C$. Because the matrix

$$\begin{bmatrix} 1 & -\Re p \\ \Im p & 1 \end{bmatrix}$$

has positive determinant we find that $\tau_{p,C} \in \mathbf{G}^+$. That $\tau_{p,C}$ satisfies (i)-(iv) is evident.

Suppose $\Re p \neq C$. Let $R = |p - C|$ and let $\theta \in (0, \pi)$ be such that $p = C + Re^{i\theta}$. Because the matrices

$$\begin{bmatrix} -\tan \frac{\theta}{2} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -(C - R) \\ 1 & -(C + R) \end{bmatrix}$$

have real entries and negative determinant we find that $\tau_{p,C} \in \mathbf{G}^+$. Keeping in mind (*) we find that, with $u = e^{i\theta}$,

$$\tau_{p,C}(z) = i \frac{1 - u}{1 + u} \frac{z - (C - R)}{z - (C + R)} \quad \text{for } z \in \mathbf{U}.$$

It is evident that (v) holds.

Suppose $v \in L$. Let $\psi \in (0, \pi)$ be such that $v = C + Re^{i\psi}$. Keeping in mind (*) we find that

$$\tau_{p,C}(v) = \frac{\tan \frac{\theta}{2}}{\tan \frac{\psi}{2}} i$$

from which (vi),(viii) and (1) follow.

Suppose $w \in \mathbf{U}$.

$$\frac{w - R}{w + R} + \frac{\overline{w} - R}{\overline{w} + R} = \frac{(w - R)(\overline{w} + R) + (\overline{w} - R)(w + R)}{|w + R|^2} = \frac{|w|^2 - R^2}{|w + R|^2}.$$

This implies that

$$\tau_{p,C}(C+w) \in \begin{cases} \mathbf{H}^+ & \text{if } |w| < R; \\ \mathbf{H}^- & \text{if } |w| > R \end{cases}$$

from which (vii) follows.

If $\sigma \in \mathbf{G}$ satisfies (i)-(iv) or (v)-(vii). Then $\tau_{p,C} \circ \sigma^{-1}$ carries i, $\mathbf{L}$ and $\mathbf{T}^+$ to themselves so must equal ι by our earlier work.

Theorem. Suppose $L \subset \mathbf{U}$, $\sigma \in \mathbf{G}$ is such that

$$L = \sigma[\mathbf{L}]$$

and $p \in L$. Then there is a unique $C \in \mathbf{R}$ such that $L = \mathbf{L}(p, C)$.
Proof. Let $q \in L \sim \{p\}$. If $\Re p = \Re q$ let $C = \Re p$; otherwise let

$$C = -\frac{|p|^2 - |q|^2}{2\Re p - 2\Re q}$$

and note that $|p - C| = |q - C|$.

Then $\tau_{p,C}^{-1} \circ \sigma$ carries the two distinct points p and q into $\mathbf{L}$ and so, by an earlier Theorem, $\tau_{p,C}^{-1} \circ \sigma[\mathbf{L}] = \mathbf{L}$. Thus

$$\mathbf{L}(p, C) = \tau_{p,C}^{-1}[\mathbf{L}] = \sigma[\mathbf{L}] = L.$$

The uniqueness of C is obvious. □
Theorem. Suppose $\sigma_i \in \mathbf{G}$ and $L_i = \sigma_i[\mathbf{L}]$, $i = 1, 2$, and $L_1 \cap L_2$ contains two or more points. Then $L_1 = L_2$.

Definition of lines, betweenness and congruence. Verification of the axioms. We say a subset L of $\mathbf{U}$ is a **Poincaré line** or **P-line** if $L = \sigma[\mathbf{L}]$ for some $\sigma \in \mathbf{G}$. It follows from our preceding work that the Incidence Axioms hold and that if L is a P-line and $p \in L$ then $L = \mathbf{L}(p, C)$ for a unique $C \in \mathbf{R}$.

Let L be a P-line. Given distinct points p, q, r on L we define q to be between p and r if

$$\Im\sigma^{-1}(p) < \Im\sigma^{-1}(q) < \Im\sigma^{-1}(r) \quad \text{for some } \sigma \in \mathbf{G} \text{ such that } L = \sigma[\mathbf{L}].$$

Previously we have shown that if $\tau \in \mathbf{G}$ and $\tau[\mathbf{L}] = \mathbf{L}$ then τ preserves the natural notion of betweenness on $\mathbf{L}$. Thus the above definition is independent of σ.

We have shown if $\tau \in \mathbf{G}$ and $\tau[\mathbf{L}] = \mathbf{L}$ then *either*

$$\frac{\tau(si)}{i} < \frac{\tau(ti)}{i} \;\Leftrightarrow\; 0 < s < t < \infty$$

or

$$\frac{\tau(si)}{i} > \frac{\tau(ti)}{i} \;\Leftrightarrow\; 0 < s < t < \infty.$$

It follows that (B1) and (B2) hold and that

(1) $$\mathbf{s}(\tau(si), \tau(ti)) = \tau[(s,t)i] \quad \text{whenever } 0 < s < t < \infty \text{ and } \tau \in \mathbf{G}.$$

Suppose $\tau \in \mathbf{G}$ and p and q be distinct points in $\mathbf{U}$. Let $\sigma \in \mathbf{G}$ be such that $\mathbf{l}(p,q) = \sigma[\mathbf{L}]$ and let $s, t \in (0, \infty)$ be such that $\sigma(s) = p$ and $\sigma(t) = q$. If $s < t$ we invoke (1) twice to obtain

$$\tau[\mathbf{s}(p,q)] = \tau[\mathbf{s}(\sigma(s), \sigma(t))] = \tau[\sigma^{-1}[(s,t)i]] = \tau \circ \sigma^{-1}[(s,t)i] = \mathbf{s}(\tau \circ \sigma^{-1}(si), \tau \circ \sigma^{-1}(ti)) = \mathbf{s}(\tau(p), \tau(q)).$$

If $s > t$ we interchange p and q and obtain the same result. Thus the members of $\mathbf{G}$ are betweenness preserving.

Suppose p and q are distinct points of $\mathbf{U} \sim \mathbf{L}$. Then $\mathbf{s}(p,q)$ meets $\mathbf{L}$ if and only if $\Re p$ and $\Re q$ have opposite signs. Thus (B3) holds for the line $\mathbf{L}$ and

$$\mathbf{H}(\mathbf{L}) = \{\mathbf{H}^+, \mathbf{H}^-\}.$$

Because a P-line is by definition the image of $\mathbf{L}$ under a member of $\mathbf{G}$ and because the members of $\mathbf{G}$ are betweenness preserving we find that (B3) holds for any P-line.

We infer from the Mapping Theorem that if $p \in \mathbf{U}$ and $C \in \mathbf{R}$ then

$$\mathbf{H}(\mathbf{L}(p,C)) = \{\mathbf{H}^+(p,C), \mathbf{H}^-(p,C)\}.$$

It follows from the Mapping Theorem and the fact that the group of $\tau \in \mathbf{G}$ which carry i to i and $\mathbf{L}$ to $\mathbf{L}$ is $\mathbf{K}$ that, given two flags $F_i = (H_i, R_i)$, $i = 1, 2$, there is exactly one member τ of $\mathbf{G}$ such that $\tau[R_1] = R_2$ and $\tau[H_1] = H_2$.