Congruence.

The Congruence Axiom.(Cong) There is a group

$$\mathcal{M}$$

of permutations of the set of points such that

$$\mathcal{M} \subset \mathcal{B}$$

and such that if $(H_i, R_i),\ i = 1, 2$, are flags then there is one and only one $\tau \in \mathcal{M}$ such that

$$\tau[H_1] = H_2 \quad \text{and} \quad \tau[R_1] = R_2.$$

We will call the members of $\mathcal{M}$ **motions**

.

Definition. Congruence of sets of points. We say the sets X and Y of points are **congruent** and write

$$X \simeq Y$$

if there exists $\tau \in \mathcal{M}$ such that $\tau[X] = Y$ in which case we say X **can be moved to** Y **(by the motion** τ**)**.

Theorem. We have

(i) If X is a set of points then $X \simeq X$.

(ii) If X and Y are sets of points and $X \simeq Y$ then $Y \simeq X$.

(iii) If X, Y and Z are sets of points, $X \simeq Y$ and $Y \simeq Z$ then $X \simeq Z$.

Remark. Equivalently, congruence is an equivalence relation on the family of sets of points.
Proof. This follows directly from the fact that the set of motions is a group. $\square$

For each set Y of points we let

$$< Y >= \{X : X \text{ is a set of points and } X \simeq Y\}$$

Corollary. We have

(i) If X is a set of points then $X \in < X >$.

(ii) If X and Y are sets of points and $X \in < Y >$ then $Y \in < X >$.

(iii) IF X, Y and Z are sets of points, $X \in < Y >$ and $Y \in < Z >$ then $X \in < Z >$.
Proof. This is an immediate consequence of the previous Theorem. $\square$

Remark. If X is a segment,line,ray,halfspace, angle, fundamental region or polygon $Y \in < X >$ then Y is of the corresponding type as well.

Definition. We let

$$\mathbf{S} = \{< S >: S \text{ is a segment}\}$$

and we let

$$\mathbf{A} = \{< A >: A \text{ is an angle}\}.$$

Note that if $S \in \mathcal{S} \in \mathbf{S}$ then S is a segment and that if $A \in \mathcal{A} \in \mathbf{A}$ then A is angle.

Theorem. Suppose a, b, c are noncollinear points and τ_i, $i = 1, 2$, are motions such that $\tau_1(a) = \tau_2(a)$, $\tau_1(b) = \tau_2(b)$ and $\tau_1(c) = \tau_2(c)$. Then $\tau_1 = \tau_2$.
Proof. Let $\sigma = \tau_2 \circ \tau_1^{-1}$ and note that $\sigma(a) = a$, $\sigma(b) = b$ and $\sigma(c) = c$. Moreover $\sigma[\mathbf{r}(a,b)] = \mathbf{r}(\sigma(a), \sigma(b)) = \mathbf{r}(a,b)$ and $\sigma[\mathbf{h}(\mathbf{l}(a,b),c)] = \mathbf{h}(\sigma[\mathbf{l}(a,b)], \sigma(c)) = \mathbf{h}(\mathbf{l}(\sigma(a),\sigma(b)),\sigma(c)) = \mathbf{h}(\mathbf{l}(a,b),c)$. Thus, by uniqueness, $\sigma = \iota$ so $\tau_1 = \tau_2$. $\square$

Reflections and half turns.

Definition. We say a permutation of a set is and **involution** if its square is the identity map of the set. One also says that such a permutation is **idempotent**.
Remark. Idempotent motions will play a fundamental role in our work.

Definition. Suppose L is a line. We say a motion ρ is a **reflection across** L if for some flag (H, R) such that $R \subset L$ we have $\rho[H] = H^{\mathbf{o}}$ and $\rho[R] = R$.

Definition. Suppose o is a point. We say a motion α is a **half turn about** o if for some flag (H, R) such that $o \in R$ we have $\rho[H] = H^{\mathbf{o}}$ and $\rho[R] = R^{\mathbf{o}}$.

Theorem. Suppose L is a line. There is exactly one reflection

$$\rho_L$$

across L. Moreover, this reflection is idempotent and fixes each point of L.
Proof. Let (H, R) be a flag such that $R \subset H$. Let ρ be that motion such that $\rho[R] = R$ and $\rho[H] = H^{\mathbf{o}}$. Then, as $\rho^2[R] = R$ and $\rho^2[H] = \rho[\rho[H]] = \rho[H^{\mathbf{o}}] = \rho[H]^{\mathbf{o}} = H^{\mathbf{oo}} = H$ we infer from uniqueness that $\rho^2 = \iota$.

Let $<$ be a geometric linear ordering of R and note that ρ preserves $<$ because $\rho[R] = R$. Suppose $p \in L$. Were it the case that $p < \rho(p)$ we would have $\rho(p) < \rho(\rho(p)) = p$ which is impossible. Were it the case that $\rho(p) < p$ we would have $p = \rho(\rho(p)) < \rho(p)$ which is impossible. Thus ρ fixes each point of L.

If S is another ray such that $S \subset L$ then $\rho[S] = S$ since ρ fixed each point of L. Thus, by uniqueness, ρ is the only reflection across L. $\square$

Theorem. Suppose o is a point. There is exactly one half turn

$$\alpha_o$$

about o. Moreover, this half turn is idempotent and carries H and R to $H^{\mathbf{o}}$ and $R^{\mathbf{o}}$, respectively, whenever (H, R) is a flag and $o \in R$.
Proof. For each flag $F = (H, R)$ such that $o \in R$ let α_F be that motion such that $\alpha_F[R] = R^{\mathbf{o}}$ and $\alpha_F[H] = H^{\mathbf{o}}$.

To proceed we need the following
Lemma. Suppose (H, R) is a flag, $o \in R$ and a is a point different from o. Then $\alpha_F^2 = \iota$ and

(1) $\qquad \alpha_F(a) \neq a \quad \text{and} \quad o \in \mathbf{s}(a, \alpha_F(a)) \quad$ whenever a is a point different from o.

Proof. We have $\alpha_F^2[H] = H$ and $\alpha_F^2[R] = R$ so $\alpha_F^2 = \iota$ by uniqueness.

If $a \in H$ then $\alpha_F(a) \in \alpha_F[H] = H^{\mathbf{o}}$ so (1) holds.

If $a \in H^{\mathbf{o}}$ then $\alpha_F(a) \in \alpha_F[H^{\mathbf{o}}] = \alpha_F[H]^{\mathbf{o}} = H^{\mathbf{oo}} = H$ so (1) holds.

Let $<$ be a geometric linear ordering of $\mathbf{l}(R)$ and note that α_F reverses $<$. If $a \in \mathbf{l}(R) \sim \{o\}$ then *either* $o < a$ in which case $\alpha_F(a) < \alpha_F(o) = o$ *or* $a < o$ in which case $o = \alpha_F(o) < \alpha_F(a)$. In either case (1) holds. $\square$

Suppose $F_i(H_i, R_i)$ are flags and $o \in R_i$, $i = 1, 2$. Suppose a is a point different from o. By the Lemma, $\alpha_{F_1}(a) \neq a$ and $o \in \mathbf{s}(a, \alpha_{F_1}(a))$. It follows that if $a \in R_2$ then $\alpha_{F_1}(a) \in R_2{}^{\mathbf{o}}$ and that if $a \in H_2$ then $\alpha_{F_1}(a) \in H_2{}^{\mathbf{o}}$. Thus $\alpha_{F_1} = \alpha_{F_2}$ by uniqueness. □

Theorem. Suppose o is a point, $\tau \in \mathcal{M}$ and $\tau(o) = o$. Then

$$\tau \circ \alpha_o = \alpha_o \circ \tau.$$

Proof. Let (H, R) be a flag such that $o = \mathbf{o}(R)$. Then

$$(\tau \circ \alpha_o)[R] = \tau[R^{\mathbf{o}}] = \tau[R]^{\mathbf{o}} \quad \text{and} \quad (\alpha_o \circ \tau)[R] = \tau[R]^{\mathbf{o}}.$$

Moreover,

$$(\tau \circ \alpha_o)[H] = \tau[H^{\mathbf{o}}] = \tau[H]^{\mathbf{o}} \quad \text{and} \quad (\alpha_o \circ \tau)[H] = \tau[H]^{\mathbf{o}}.$$

So $\tau \circ \alpha_o = \alpha_o \circ \tau$ by uniqueness. □

Theorem. Suppose A is an angle with vertex o and B is the angle vertical to A. Then the half turn about o interchanges A and B. In particular, $A \simeq B$.
Proof. Let H and I be halfspaces such that $A = H \cap I$. Then $\tau[A] = \tau[H] \cap \tau[I] = H^{\mathbf{o}} \cap I^{\mathbf{o}} = B$ and $A = \tau^2[A] = \tau[B]$. □

Theorem. Suppose A_i, B_i, $i = 1, 2$, are angles, B_i is supplementary to A_i, $i = 1, 2$, and $A_1 \simeq A_2$. Then $B_1 \simeq B_2$.
Proof. Let H_i, I_i, $i = 1, 2$ be halfspaces such that $A_i = H_i \cap I_i$ and $B_i = H_i \cap I_i{}^{\mathbf{o}}$, $i = 1, 2$. Let $\tau \in \mathcal{M}$ be such that $\tau[A_1] = A_2$. Since $\tau[H_1] \cap \tau[I_1] = \tau[A_1] = A_2 = H_2 \cap I_2$ we infer from earlier work that *either* $\tau[H_1] = H_2$ and $\tau[I_1] = I_2$ *or* $\tau[H_1] = I_2$ and $\tau[I_1] = H_2$. In the first case we have

$$\tau[B_1] = \tau[H_1 \cap I_1{}^{\mathbf{o}}] = \tau[H_1] \cap \tau[I_1]^{\mathbf{o}} = H_2 \cap I_2{}^{\mathbf{o}} = B_2.$$

In the second case we have

$$\tau[B_1] = \tau[H_1 \cap I_1{}^{\mathbf{o}}] = \tau[H_1] \cap \tau[I_1]^{\mathbf{o}} = I_2 \cap H_2{}^{\mathbf{o}}$$

which is vertical to B_2 and, by virtue of the preceding Theorem, congruent to B_2. □

Theorem. Suppose τ is a motion, $\tau \neq \iota$ and $\tau^2 = \iota$. Then τ is either a half turn or a reflection.
Proof. Since $\tau \neq \iota$ there is a point a such that $\tau(a) \neq a$. Let $L = \mathbf{l}(a, \tau(a))$ and note that $\tau[L] = L$. Let $H \in \mathbf{H}(L)$.

Case One. $\tau[H] = H^{\mathbf{o}}$. Choose $b \in H$. Then $\mathbf{s}(b, \tau(b))$ meets L in a point o. Let $M = \mathbf{l}(b, \tau(b))$. Then $\tau[M] = M$ so $\tau(o) \in \tau[L] \cap \tau[M] = L \cap M$. Since $L \neq M$ we have $L \cap M = \{o\}$ so $\tau(o) = o$. Let $<$ be a geometric linear ordering on L such that $a < \tau(a)$. Then $\tau(\tau(a)) = a < \tau(a)$ so τ is reverses $<$ on L. It follows that if $R = \{x \in L : o < x\}$ then $\tau[R] = R^{\mathbf{o}}$. Thus, by an earlier Theorem, τ is the half turn about o.

Case Two. $\tau[H] = H$. We will show that there is $p \in H$ such that $\tau(p) = p$.

Let $R = \mathbf{r}(a, \tau(a))^{\mathbf{o}}$ and let $\prec$ be the geometric linear ordering on $\mathbf{A}(H, R)$.

Suppose $b \in H$ and $\tau(b) \neq b$. Let $K = \mathbf{l}(b, \tau(b))$ note that $\tau[K] = K$. It is impossible that $\mathbf{r}(a, b) = \mathbf{r}(a, \tau(b))$ since that would force the points $a, b, \tau(b)$ to lie on K which in turn would force $\tau(a) \in \tau[K]$ which would imply $K = L$ which is impossible. Choose $c \in \{b, \tau(b)\}$ such that if $S = \mathbf{r}(a, c)$ and $T = \mathbf{r}(a, \tau(c))$ then $S \prec T$. Let $M = \mathbf{l}(a, \tau(c))$ and let $I \in \mathbf{H}(M)$ be such that $\mathbf{a}(R, T) = H \cap I$. Then $S \subset \mathbf{a}(R, T) \subset I$ and $R = I \cap \mathbf{b}(H) \subset I$ so c is on the same side of M as R. Since $\tau(a) \in R^{\mathbf{o}}$ we find that c and $\tau(a)$ are on different sides of I. Thus $\mathbf{s}(c, \tau(a))$ and therefore $N = \mathbf{l}(c, \tau(a))$ meets M in a point p. Since $\tau(p) \in \tau[M \cap N] = \tau[M] \cap \tau[N] = N \cap M = M \cap N$ and since $M \neq N$ we find that $\tau(p) = p$.

Interchanging H with $H^{\mathbf{o}}$ above we infer the existence of $q \in H^{\mathbf{o}}$ such that $\tau(q) = q$.

Let $O = \mathbf{l}(p,q)$ and let $Q = \mathbf{r}(p,q)$. Then $\tau[Q] = \mathbf{r}(\tau(p),\tau(q)) = \mathbf{r}(p,q) = Q$ so, by an earlier Theorem, τ is reflection across O. $\square$

Two fundamental consequences of the Continuity Axiom.

In order to proceed further we need the following two Theorems the proof which uses the Continuity Axiom. If that bothers you, and I can see how it could, you may assume these Theorems as axioms. We will not refer to the Continuity Axiom for the remainder of this Chapter.

Theorem. (CS) Suppose R is a ray with origin o, $p,q \in R$ and $\mathbf{s}(o,p) \simeq \mathbf{s}(o,q)$. Then $p = q$.
Proof. Choose a motion τ such that $\tau[\mathbf{s}(o,p)] = \mathbf{s}(o,q)$. Since a motion carries the endpoints of a segment to the endpoints of a segment we find that *either* $\tau(o) = o$ and $\tau(p) = q$ *or* $\tau(o) = q$ and $\tau(p) = o$.

Let $L = \mathbf{l}(o,p)$.

Suppose $\tau(o) = o$ and $\tau(p) = q$. Then $\tau[R] = \tau[\mathbf{r}(o,p)] = \mathbf{r}(\tau(o),\tau(p)) = \mathbf{r}(o,q) = R$. By an earlier Theorem, τ is idempotent so $q = \tau(p) = p$.

Suppose $\tau(o) = q$ and $\tau(p) = o$. Let $<$ be the geometric linear ordering of L. Since $\tau(p) = o < q = \tau(o)$ we find that τ is reverses $<$. Thus, by the Continuity Axiom and earlier work, there is a point $r \in L$ such that $\tau(r) = r$. Let $S = \{x \in L : r < x\}$ and note that $\tau[S] = S^{\mathbf{o}}$. Since τ^2 preserves the members of $\mathbf{H}(L)$ and carries S to S we infer from uniqueness that $\tau^2 = \iota$. Thus $p = \tau(\tau(p)) = \tau(o) = q$. $\square$

Theorem.(CA) Suppose (H,R), $S,T \in \mathbf{A}(H,R)$ and $\mathbf{a}(R,S) \simeq \mathbf{a}(R,T)$. Then $S = T$.
Proof. Let τ be a motion such that $\tau[\mathbf{a}(R,S)] = \mathbf{a}(R,T)$. Then *either* $\tau[R] = R$ and $\tau[S] = T$ *or* $\tau[R] = T$ and $\tau[S] = R$.

Suppose $\tau[R] = R$ and $\tau[S] = T$. As $S \subset H$ and $\tau[S] = T \subset H$ we infer that $\tau[H] = H$. From uniqueness we infer that $\tau = \iota$ so $S = \tau[S] = T$.

Suppose $\tau[R] = T$ and $\tau[S] = R$. Then $\tau[R] = T \subset H$ and $R = \tau[S] \subset \tau[H]$. By the Continuity Axiom and earlier work there is a ray U such that $\tau[U] = U$. By an earlier Theorem, either $\tau = \iota$ or $\tau^2 = \iota$. Since $\tau[R] = S \subset H$ and $R \cap H = \emptyset$ we find that $\tau^2 = \iota$. Thus $S = \tau[\tau[S]] = \tau[R] = T$. $\square$

Some fundamental theorems of neutral geometry.

The following Theorem is a direct consequence of (CS). It amounts to one of the congruence axioms in the book.
Theorem. Suppose S is a segment and R is a ray with origin o. There is one and only one $p \in R$ such that $S \simeq \mathbf{s}(o,p)$.
Proof. The uniqueness of such a point p follows from (CS). To prove existence, let a,b be points such that $S = \mathbf{s}(a,b)$, let G be a halfspace such that $\mathbf{b}(G) = \mathbf{l}(a,b)$ and let H be a halfspace such that $\mathbf{b}(H) = \mathbf{l}(R)$. Let τ be the motion such that $\tau[\mathbf{r}(a,b)] = R$ and $\tau[G] = H$. Let $p = \tau(b)$. $\square$

The following Theorem is a direct consequence of (CA). It amounts to one of the congruence axioms in the book.
Theorem. Suppose A is an angle and (H,R) is a flag. There is one and only one $S \in \mathbf{A}(H,R)$ such that $A \simeq \mathbf{a}(R,S)$.
Proof. The uniqueness of the ray S follows from (CA). To prove existence, let P,Q be rays such that $A = \mathbf{a}(P,Q)$. Let τ be the motion such that $\tau[\mathbf{h}(\mathbf{l}(P),Q)] = H$ and $\tau[P] = R$. Let $S = \tau[Q]$. $\square$

Definition. Suppose a,b are distinct points. We say the motion τ **transposes** a **and** b if $\tau(a) = b$ and $\tau(b) = a$

Suppose R and S are distinct rays with a common origin. We say the motion τ **transposes** R **and** S if $\tau[R] = S$ and $\tau[S] = R$.

Theorem. Suppose a,b are distinct points. There are exactly two motions which carry $\mathbf{r}(a,b)$ to $\mathbf{r}(b,a)$. Each transposes a and b. One is a half turn and the other is a reflection.

Proof. Let $<$ be the linear ordering of $L = \mathbf{l}(a,b)$ such that $a < b$; let $R = \mathbf{r}(a,b) = \{x \in L : a < x\}$; let $S = \mathbf{r}(b,a) = \{x \in L : x < b\}$; let $H \in \mathbf{H}(L)$; and let $\mathcal{T}$ be the set of motions which carry R to S.

Let τ be the motion such that $\tau[R] = S$ and $\tau[H] = H$ and let $\tau^{\mathbf{o}}$ be the motion such that $\tau[R] = S$ and $\tau[H] = H^{\mathbf{o}}$. Thus $\{\tau, \tau^{\mathbf{o}}\} \subset \mathcal{T}$.

Suppose $\sigma \in \mathcal{T}$.

Since a is the origin of R and b is the origin of S we find that $\sigma(a) = b$ and $\sigma[R^{\mathbf{o}}] = S^{\mathbf{o}}$. Thus $\sigma[L] = L$ so $\sigma[H] \in \{H, H^{\mathbf{o}}\}$. It follows by uniqueness that there either $\sigma = \tau$ of $\sigma = \tau^{\mathbf{o}}$. Thus $\mathcal{T}$ has exactly two members. Now $\sigma[\mathbf{s}(a,b)] = \mathbf{s}(\sigma(a), \sigma(b)) = \mathbf{s}(b, \sigma(b))$ so $\mathbf{s}(b,a) = \mathbf{s}(a,b) \simeq \mathbf{s}(b, \sigma(b))$. Since $b \in R$ we find that $\sigma(b) \in \sigma[R] = S$. Also, $a \in S$. We infer from (CS) that $\sigma(b) = a$. Thus σ transposes a and b.

Evidently, $\sigma^2[H] = H$. Since $\sigma(b) = a < b = \sigma(a)$ we infer that σ reverses $<$. Thus $\sigma^2[R] = \sigma[S] = \{x \in L : a < x\} = R$. Thus $\sigma^2 = \iota$ by uniqueness. □

Theorem. Suppose R and S are rays with a common origin; $H = h\mathbf{l}(R), S$ and $I = \mathbf{h}(\{LS, R)$; and τ is the motion such that $\tau[R] = S$ and $\tau[H] = I$. Then τ tranposes R and $S.\tau^2 = \iota$.

Moreover, τ is a reflection and is the only motion which transposes R and S.

Proof. Since $R \subset I$ and, as $S \subset H$, we have $\tau[S] \subset \tau[H] = I$. Moreover, $\mathbf{a}(S, \tau[S]) = \mathbf{a}(\tau[R], \tau[S]) \simeq \tau[\mathbf{a}(R,S)] = \tau[\mathbf{a}(S,R)]$. Thus, by (CA), $\tau[S] = R$. In particular, $\tau[I] \in \{H, H^{\mathbf{o}}\}$. Now $S = \tau[R] \subset \tau[I]$ as $R \subset I$. Since $S \subset H$ we have $\tau[I] = H$. It follows from uniqueness that $\tau^2 = \iota$.

Suppose σ is a motion which transposes R and S. Then $\sigma[H] \in \{I, I^{\mathbf{o}}\}$ and $\sigma[I] \in \{H, H^{\mathbf{o}}\}$. Since $R = \sigma[S] \subset \sigma[H]$ and $R \subset I$ we find that $\sigma[H] = I$. Thus, by uniqueness, $\sigma = \tau$. □

Theorem. (SAS) Suppose T_i is a triangle with vertices a_i, b_i, c_i, $i = 1, 2$. Suppose

$$\mathbf{s}(a_1, b_1) \simeq \mathbf{s}(a_2, b_2), \quad \mathbf{a}(b_1, a_1, c_1) \simeq \mathbf{a}(b_2, a_2, c_2), \quad \mathbf{s}(a_1, c_1) \simeq \mathbf{s}(a_2, c_2).$$

Then there is a motion τ such that

$$\tau(a_1) = a_2, \quad \tau(b_1) = b_2, \quad \tau(c_1) = c_2.$$

Moreover $\tau[T_1] = T_2$.

Proof. Since $\mathbf{a}(b_1, a_1, c_1) \simeq \mathbf{a}(b_2, a_2, c_2)$ we may use the existence of motions which transpose rays with a common origin to obtain a motion τ such that $\tau[\mathbf{r}(a_1, b_1)] = \mathbf{r}(a_2, b_2)$ and $\tau[\mathbf{r}(a_1, c_1)] = \mathbf{r}(a_2, c_2)$. Then $\tau(a_1) = a_2$ since τ carries the origin a_1 of $\mathbf{r}(a_1, b_1)$ to the origin a_2 of $\mathbf{r}(a_2, b_2)$. Now $\tau(b_1) \in \tau[\mathbf{r}(a_1, b_1)] = \mathbf{r}(a_2, b_2)$ and $\tau(c_1) \in \tau[\mathbf{r}(a_1, c_1)] = \mathbf{r}(a_2, c_2)$. Applying (CS) twice we infer that $\tau(b_1) = b_2$ and $\tau(c_1) = c_2$. By earlier work we infer that $\tau[T_1] = T_2$. □

Theorem. (ASA) Suppose T_i is a triangle with vertices a_i, b_i, c_i, $i = 1, 2$. Suppose

$$\mathbf{a}(c_1, a_1, b_1) \simeq \mathbf{a}(c_2, a_2, b_2), \quad \mathbf{s}(a_, b_1) \simeq \mathbf{s}(a_2, b_2), \quad \mathbf{a}(c_1, b_1, a_1) \simeq \mathbf{a}(c_2, b_2, a_2).$$

Then there is a motion τ such that

$$\tau(a_1) = a_2, \quad \tau(b_1) = b_2, \quad \tau(c_1) = c_2.$$

Moreover $\tau[T_1] = T_2$.

Proof. Since $\mathbf{a}(c_1, a_1, b_1) \simeq \mathbf{a}(c_2, a_2, b_2)$ we may use the existence of motions which transpose rays with a common origin to obtain a motion τ such that $\tau[\mathbf{r}(a_1, b_1)] = \mathbf{r}(a_2, b_2)$ and $\tau[\mathbf{r}(a_1, c_1)] = \mathbf{r}(a_2, c_2)$. Then $\tau(a_1) = a_2$ since τ carries the origin a_1 of $\mathbf{r}(a_1, b_1)$ to the origin a_2 of $\mathbf{r}(a_2, b_2)$. Since $\tau(b_1) \in \tau[\mathbf{r}(a_1, b_1)] = \mathbf{r}(a_2, b_2)$ we may use (CS) to infer that $\tau(b_1) = b_2$.

Next note that $\tau(c_1) \in \tau[\mathbf{r}(a_1, c_1)] = \mathbf{r}(a_2, c_2) \subset \mathbf{h}(\mathbf{l}(a_2, b_2), c_2)$. Moreover,

$$\mathbf{a}(a_2, b_2, c_2) \simeq \mathbf{a}(a_1, b_1, c_1) \simeq \tau[\mathbf{a}(a_1, b_1, c_1)] = \mathbf{a}(\tau(a_1), \tau(b_1), \tau(c_1)) = \mathbf{a}(a_1, b_2, \tau(c_1)).$$

By (CA) we infer that $\mathbf{r}(b_2, c_2) = \mathbf{r}(b_2, \tau(c_1))$. Thus $\tau(c_1) \in \mathbf{l}(b_2, c_2) \cap \mathbf{l}(a_2, c_2)$ and thus must equal c_2. By earlier work we infer that $\tau[T_1] = T_2$. □

Alternate interior angle theorem. Suppose L, M are lines; $a \in L \sim M$ and $b \in M \sim L$; $H = \mathbf{h}(L, b)$ and $I = \mathbf{h}(M, a)$; $N = \mathbf{l}(a, b)$ and $J \in \mathbf{H}(M)$; and

$$H \cap J \simeq I \cap J^{\mathbf{o}}.$$

Then L and M are parallel.
Remark. The pair of angles $H \cap J$ and $I \cap J^{\mathbf{o}}$ are called **alternate interior angles**.
Proof. Let $R = \mathbf{r}(a, b) = H \cap N$, let $S = \mathbf{r}(b, a) = I \cap N$, note that $H \cap J = \mathbf{a}(H \cap N, L \cap J) = \mathbf{a}(R, L \cap J)$ and note that $I \cap J^{\mathbf{o}} = \mathbf{a}(I \cap N, M \cap J^{\mathbf{o}}) = \mathbf{a}(S, M \cap J^{\mathbf{o}})$. Owing to the existence of motions which transpose rays with a common origin we obtain a motion such that $\tau[R] = S$ and $\tau[L \cap J] = M \cap J^{\mathbf{o}}$. In particular, $\tau[L] = M$ and $\tau[J] = J^{\mathbf{o}}$. Since $\tau[R] = S$ we infer from earlier work that τ transposes a and b and that $\tau^2 = \iota$. Thus $\tau[S] = R$ and $\tau[M] = L$.

Suppose, contrary to the Theorem, there were a point $c \in L \cap M$. Note that $c \notin \{a, b\}$. Now $\tau(c) \in \tau[L \cap M] = \tau[L] \cap \tau[M] = M \cap L$. Thus $\tau(c) = c$ since otherwise the unequal lines L and M would have two points in common. The point c cannot be in either J or $J^{\mathbf{o}}$ since τ interchanges J and $J^{\mathbf{o}}$. Thus $c \in N$. Thus $L = \mathbf{l}(a, c) = N = \mathbf{l}(b, c) = M$ which is impossible. $\square$

Corollary. Suppose L is a line and $p \notin L$. There is a line passing through p parallel to L.
Remark. In Euclidean geometry this parallel is unique. In hyperbolic geometry it is not.

Theorem. (SAA) Suppose T_i is a triangle with vertices a_i, b_i, c_i, $i = 1, 2$. Suppose

$$\mathbf{a}(c_1, a_1, b_1) \simeq \mathbf{a}(c_2, a_2, b_2), \quad \mathbf{s}(a_, b_1) \simeq \mathbf{s}(a_2, b_2), \quad \mathbf{a}(a_1, c_1, b_1) \simeq \mathbf{a}(a_2, c_2, b_2).$$

Then there is a motion τ such that

$$\tau(a_1) = a_2, \quad \tau(b_1) = b_2, \quad \tau(c_1) = c_2.$$

Moreover $\tau[T_1] = T_2$
Proof. Since $\mathbf{a}(c_1, a_1, b_1) \simeq \mathbf{a}(c_2, a_2, b_2)$ we may use the existence of motions which transpose rays with a common origin to obtain a motion τ such that $\tau[\mathbf{r}(a_1, b_1)] = \mathbf{r}(a_2, b_2)$ and $\tau[\mathbf{r}(a_1, c_1)] = \mathbf{r}(a_2, c_2)$. Then $\tau(a_1) = a_2$ since τ carries the origin a_1 of $\mathbf{r}(a_1, b_1)$ to the origin a_2 of $\mathbf{r}(a_2, b_2)$. Since $\tau(b_1) \in \tau[\mathbf{r}(a_1, b_1)] = \mathbf{r}(a_2, b_2)$ we may use (CS) to infer that $\tau(b_1) = b_2$.

Let $N = \mathbf{l}(a_2, c_2)$. We have $\tau(c_1) \in \mathbf{r}(a_2, c_2) \subset N$. We have

$$\mathbf{a}(a_2, c_2, b_2) \simeq \tau[\mathbf{a}(a_1, b_1, c_1)] = \mathbf{a}(\tau(a_1), \tau(b_1), \tau(c_1)) = \mathbf{a}(a_2, b_2, \tau(c_1)).$$

Were it the case that $\tau(c_1) \neq c_2$ we would infer from the congruence of vertical angles and the alternate interior angle theorem that $\mathbf{l}(b_2, c_2)$ is parallel to $\mathbf{l}(b_2, \tau(c_1))$ which is absurd. $\square$

Right angles and perpendiculars.

Definition. We say an angle is a **right angle** if it is congruent to one of its supplements. We say the lines L and M are **perpendicular** if $L \cap M$ contains exactly one point and there are $H \in \mathbf{H}(L)$ and $I \in \mathbf{H}(M)$ such that $H \cap I$ is a right angle; it follows from the foregoing that $H \cap I^{\mathbf{o}}$, $H^{\mathbf{o}} \cap I$ and $H^{\mathbf{o}} \cap I^{\mathbf{o}}$ are right angles.

Theorem. An angle congruent to a right angle is a right angle.
Proof. Simple exercise for the reader. $\square$

Theorem. Suppose (H, R) is a flag. There is exactly one ray $S \in \mathbf{A}(H, R)$ such that $\mathbf{a}(R, S)$ is a right angle.

Moreover, if $L = \mathbf{l}(R)$, $M = \mathbf{l}(S)$ and o is the origin of R then

$$\rho_L \circ \rho_M = \alpha_o = \rho_M \circ \rho_L.$$

Proof. Let o be the origin of R. Let τ be that motion such that $\rho[H] = H$ and $\rho[R] = R^{\mathbf{o}}$. Then $\rho(o) = o$ and $\rho^2 = \iota$ by uniqueness. By a previous Theorem, ρ is either a reflection or a half turn. Since $\rho(o) = o$ and $\rho[H] = H$ it cannot be a half turn so it must be a reflection. . Let M be that line such that ρ is reflection across M. Note that $L \cap M = \{o\}$. Let $S = H \cap M$. We have $\rho[S] = \rho[H] \cap \rho[M] = H \cap M = S$ so $\rho[\mathbf{a}(R,S)] = \mathbf{a}(R^{\mathbf{o}}, S)$ so $\mathbf{a}(R,S)$ is a right angle.

Suppose $T \in \mathbf{A}(H,R)$ and $\mathbf{a}(R,T)$ is a right angle $T \subset \mathbf{a}(R,S)$.Since $\mathbf{a}(R,T) \simeq \mathbf{a}(R^{\mathbf{o}},T)$ and since there exists a motion which transposes the sides of a given angle there is a motion σ such that $\sigma[R] = R^{\mathbf{o}}$ and $\sigma[T] = T$. Since $T \cup \sigma[T] \subset H$ we have $\sigma[H] \subset H$. Thus $\sigma = \rho$ by uniqueness so $\rho[T] = T$. Since ρ is reflection across M we have $\mathbf{l}(T) = M$ so $T = S$. □

Corollary. Any two right angles are congruent.
Proof. Simple exercise for the reader. □

Theorem. Suppose M and N are lines perpendicular to the same line L. Then $L = M$ or L is parallel to M.
Proof. Use the preceding Corollary and the alternating interior angle theorem. □

Theorem. Suppose L is a line and $b \notin L$. There is one and only one point $a \in L$ such that $\mathbf{l}(a,b)$ is perpendicular to L.
Proof. Since $\rho_L(b) \in \mathbf{h}(L,b)^{\mathbf{o}}$ there is a point in $L \cap \mathbf{s}(a, \rho_L(a))$. We leave as a simple exercise for the reader the proof that $\mathbf{l}(a, \rho_L(a))$ is perpendicular to L.

The uniqueness of a follows directly from the preceding Theorem.
Remark. The point a is said to be **at the foot of the perpendicular dropped from** b **to** a.

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Segment arithmetic and ordering.

Lemma. Suppose a_i, c_i are distinct points and $b_i \in \mathbf{s}(a_i, c_i)$, $i = 1, 2$, and suppose

$$\mathbf{s}(a_1, b_i) \simeq \mathbf{s}(a_2, b_2).$$

Then

$$\mathbf{s}(b_1, c_1) \simeq \mathbf{s}(b_2, c_2) \ \Leftrightarrow\ \mathbf{s}(a_1, c_1) \simeq \mathbf{s}(a_2, c_2).$$

Proof. Making use of the fact that there exist motions which interchange any pair of points we infer the existence of a motion σ such that $\sigma(a_1) = a_2$ and $\sigma(b_1) = b_2$

Suppose $\mathbf{s}(b_1, c_1) \simeq \mathbf{s}(b_2, c_2)$. Then $\mathbf{s}(b_2, \tau(c_1)) = \mathbf{s}(\tau(b_1), \tau(c_1)) \simeq \mathbf{s}(b_2, \tau(c_1))$ so $\tau(c_1) = c_2$ by (AC) and $\mathbf{s}(a_1, c_1) \simeq \mathbf{s}(a_2, c_2)$.

Suppose $\mathbf{s}(a_1, c_1) \simeq \mathbf{s}(a_2, c_2)$. Then $\mathbf{s}(a_2, \tau(c_1)) = \mathbf{s}(\tau(a_1), \tau(c_1)) \simeq \mathbf{s}(a_2, \tau(c_1))$ so $\tau(c_1) = c_2$ by (AC) and $\mathbf{s}(b_1, c_1) \simeq \mathbf{s}(b_2, c_2)$. □

Definition. Whenever m is a positive integer and $\mathcal{S}_i \in \mathbf{S},\ i = 1, \ldots, m$, we let

$$+(\mathcal{S}_1, \ldots, \mathcal{S}_m)$$

be the set of ordered m-tuples $(x_0, \ldots, x_m)$ such that there are a line L and a geometric linear ordering $<$ on L such that $x_i \in L,\ i = 1, \ldots, m$; $x_0 < x_1 < \ldots < x_m$; and $\mathbf{s}(x_{i-1}, x_i) \in \mathcal{S}_i,\ i = 1, \ldots, m$.

Lemma. Suppose m is a positive integer and $\mathcal{S}_i \in \mathbf{S},\ i = 1, \ldots, m$.
Then

$$\{\mathbf{s}(x_0, x_m) : (x_0, \ldots, x_m) \in\ +(\mathcal{S}_1, \ldots, \mathcal{S}_m)\} \in \mathbf{S}.$$

Proof. We will prove this for $m = 2$. A straightforward induction will handle the other cases.

Suppose $\mathcal{S}, \mathcal{T} \in \mathbf{S}$. Let a, b be distinct points such that $S = \mathbf{s}(a,b)$. Let c be that point in $\mathbf{r}(b,a)^{\mathbf{o}}$ such that $T \simeq \mathbf{s}(b,c)$. Then $(a,b,c) \in +(\mathcal{S}, \mathcal{T})$.

Suppose $(a_i, b_i, c_i) \in +(\mathcal{S}, \mathcal{T}),\ i = 1, 2$. From the preceding Lemma we infer that $\mathbf{s}(a_1, c_1) \simeq \mathbf{s}(a_2, c_2)$. $\square$

Definition. For each $\mathcal{S}, \mathcal{T} \in \mathbf{S}$ we let

$$\mathcal{S} + \mathcal{T} = \{\mathbf{s}(a, c) : (a, b, c) \in +(\mathcal{S}, \mathcal{T})\}$$

and note that, by virtue of the previous Theorem,

$$\mathcal{S} + \mathcal{T} \in \mathbf{S}.$$

Theorem. Suppose $\mathcal{S}, \mathcal{T}, \mathcal{U} \in \mathbf{S}$. Then

$$\mathcal{S} + \mathcal{T} = \mathcal{T} + \mathcal{S}; \tag{1}$$

$$(\mathcal{S} + \mathcal{T}) + \mathcal{U} = \mathcal{S} + (\mathcal{T} + \mathcal{U}); \tag{2}$$

$$\mathcal{S} + \mathcal{U} = \mathcal{T} + \mathcal{U} \ \Rightarrow\ \mathcal{S} = \mathcal{T}. \tag{3}$$

$$\mathcal{S} + \mathcal{T} \neq \mathcal{S}. \tag{4}$$

Proof. Suppose If $(a, b, c) \in +(\mathcal{S}, \mathcal{T})$ then $(c, b, a) \in +(\mathcal{S}, \mathcal{T})$ so (1) holds.

Suppose $(a, b, c, d) \in +(\mathcal{S}, \mathcal{T}, \mathcal{U})$. Then $(a, b, c) \in +(\mathcal{S}, \mathcal{T})$ so $(a, c, d) \in +(\mathcal{S} + \mathcal{T}, \mathcal{U})$ so $\mathbf{s}(a, d) \in (\mathcal{S} + \mathcal{T}) + \mathcal{U}$. Moreover, $(b, c, d) \in +(\mathcal{T}, \mathcal{U})$ so $(a, b, d) \in +(\mathcal{S}, \mathcal{T} + \mathcal{U})$ so $\mathbf{s}(a, d) \in \mathcal{S} + (\mathcal{T} + \mathcal{U})$. (2) follows.

Suppose $\mathcal{S} + \mathcal{U} = \mathcal{T} + \mathcal{U}$. Let $(a_1, b_1, c_1) \in +(\mathcal{S}, \mathcal{U})$ and let $(a_2, b_2, c_2) \in +(\mathcal{T}, \mathcal{U})$. By the Lemma in the beginning of this section we infer that $\mathbf{s}(a_1, b_1) \simeq \mathbf{s}(a_2, b_2)$ so (3) holds.

Let $(a, b, c) \in +(\mathcal{S}, \mathcal{T})$. Then $\mathbf{s}(a, c) \in \mathcal{S} + \mathcal{T}$ and $\mathbf{s}(a, b) \in \mathcal{S}$. Were it the case that $\mathcal{S} + \mathcal{T} = \mathcal{S}$ we would have that $\mathbf{s}(a, c) \simeq \mathbf{s}(a, b)$ which is excluded by (CS). $\square$

Corollary. Suppose S, T are segments, $S \subset T$ and $S \neq T$. Then there is $\mathcal{U} \in \mathbf{S}$ such that

$$< T > = < S > + \mathcal{U}.$$

Proof. Let L be the line containing T, let a, d be distinct points in L such that $T = \mathbf{s}(a, d)$ and let $<$ be the geometric linear ordering of L such that $a < d$. Let b, c be distinct points in L such that $S = \mathbf{s}(b, c)$.

If $a < b$ and $c = d$ then $< T > = < \mathbf{s}(a, b) > + < S > = < S > + < \mathbf{s}(a, b) >$. If $a = b$ and $c < d$ then $< T > = < S > + < \mathbf{s}(c, d) >$. If $a < b$ and $b < d$ then $< T > = < \mathbf{s}(a, b) > + < S > + < \mathbf{s}(c, d) > = < S > +(< \mathbf{s}(a, b) > + < \mathbf{s}(c, d) >)$. $\square$

Definition. Suppose $\mathcal{S}, \mathcal{T} \in \mathbf{S}$. We write

$$\mathcal{S} < \mathcal{T}$$

and say $\mathcal{S}$ **is less than** $\mathcal{T}$ if there is are $S \in \mathcal{S}$ and $T \in \mathcal{T}$ such that $S \subset T$ and $S \neq T$.

Remark. The following Theorem implies that $<$ induces a linear ordering on $\mathbf{S}$.

Theorem. If $\mathcal{S}, \mathcal{T} \in \mathbf{S}$ then exactly one of the following holds:

$$\mathcal{S} < \mathcal{T}; \quad \mathcal{S} = \mathcal{T}; \quad \mathcal{T} < \mathcal{S}.$$

If $\mathcal{S}, \mathcal{T}, \mathcal{U} \in \mathbf{S}$ then

$$\mathcal{S} < \mathcal{T} \quad \text{and} \quad \mathcal{T} < \mathcal{U} \ \Rightarrow\ \mathcal{S} < \mathcal{U}.$$

If $\mathcal{S}, \mathcal{T} \in \mathbf{S}$ then
$$\mathcal{S} < \mathcal{T} \;\Leftrightarrow\; \text{there is } \mathcal{U} \in \mathbf{S} \text{ such that } \mathcal{S} + \mathcal{U} = \mathcal{T}$$

Proof. Exercise using the previous Theorems. □

Definition. We say the point o is a **midpoint of the segment** S if $\mathbf{s}(a,o) \simeq \mathbf{s}(o,b)$ whenever a, b are distinct points such that $S = \mathbf{s}(a,b)$.

Definition. For each positive integer m and each $\mathcal{S} \in \mathbf{S}$ we define

$$n\mathcal{S}$$

inductively by letting $1\mathcal{S} = \mathcal{S}$ and requiring that $(n+1)\mathcal{S} = n\mathcal{S} + \mathcal{S}$.

Theorem. Suppose $\mathcal{S} \in \mathbf{S}$. Then
$$(m+n)\mathcal{S} = m\mathcal{S} + n\mathcal{S};$$
$$m(n\mathcal{S}) = (mn)\mathcal{S};$$
$$m\mathcal{S} = n\mathcal{S} \;\Rightarrow\; m = n;$$
$$m < n \;\Leftrightarrow\; m\mathcal{S} < n\mathcal{S}.$$

Proof. Simple exercise making use of induction and the foregoing. □

Theorem. Suppose $\mathcal{S}, \mathcal{T} \in \mathbf{S}$ and m is a positive integer such that

$$m\mathcal{S} = m\mathcal{T}.$$

Then $\mathcal{S} = \mathcal{T}$.
Proof. The Theorem holds trivially in case $m = 1$ so suppose $m > 1$. Suppose, contrary to the Theorem, $\mathcal{S} \neq \mathcal{T}$. Then it either $\mathcal{S} < \mathcal{T}$ in which case there is $\mathcal{U} \in \mathbf{S}$ such that $\mathcal{T} = \mathcal{S} + \mathcal{U}$ *or* $\mathcal{T} < \mathcal{S}$ in which case there is $\mathcal{U} \in \mathbf{S}$ such that $\mathcal{S} = \mathcal{T} + \mathcal{U}$. In the first case we have

$$m\mathcal{S} = m\mathcal{S} = m\mathcal{S} + \mathcal{U} = m\mathcal{S} + m\mathcal{U};$$

cancelling $(m-1)\mathcal{S}$ from both sides we obain

$$\mathcal{S} = \mathcal{S} + m\mathcal{U}$$

which is impossible. The case $\mathcal{T} < \mathcal{S}$ may be handled the same way. □

Definition. Suppose n is a positive integer and $\mathcal{T} \in \mathbf{S}$. Keeping in mind the previous Theorem we let

$$\frac{\mathcal{S}}{n}$$

to be that member of $\mathbf{S}$ such that, if it exists, is such that

$$n\left(\frac{\mathcal{S}}{n}\right) = \mathcal{S}.$$

Theorem. Every segment has a unique midpoint.
Proof. The existence of midpoints of segments has already been established.

Let a, b be distinct points; let $S = \mathbf{s}(a,b)$; and let o and p be midpoints of S. Let $\mathcal{S} =< \mathbf{s}(a,o) >$ and let $\mathcal{T} =< \mathbf{s}(a,p) >$. Then $2\mathcal{S} = 2\mathcal{T}$ so, by the previous Theorem, $\mathcal{S} = \mathcal{T}$. That $o = p$ follows from (CS). □

Angle arithmetic and ordering.

Definition. An **angle measure** is an ordered pair

$$(m, \mathcal{A})$$

such that m is a nonnegative integer and *either* $\mathcal{A} \in \mathbf{A}$ *or* $\mathcal{A} =< \emptyset >$.

Theorem. Suppose (H, R) is a flag with origin o; $S, T \in \mathbf{R}(o)$;

$$S \subset H; \quad T \subset \mathbf{h}(\mathbf{l}(S), R)^{\mathbf{o}};$$

and

$$C = \mathbf{a}(R, S) \cup S \cup \mathbf{a}(S, T).$$

Then
either
$T \subset H$ and $C = \mathbf{a}(R, T)$
or
$T = R^{\mathbf{o}}$ and $C = H$
or
$T \subset H^{\mathbf{o}}$ and $C = H \cup R^{\mathbf{o}} \cup \mathbf{a}(R^{\mathbf{o}}, T)$.
Proof. Exercise. □

Theorem. Suppose $(m, \mathcal{A})$ and $(n, \mathcal{B})$ are angle measures. There is one and only one angle measure

$$(m, \mathcal{A}) + (n, \mathcal{B})$$

such that

$$(m, \mathcal{A}) + (n, \mathcal{B}) = \begin{cases} (m + n, \mathbf{a}(R, T)) & \text{if } T \subset H; \\ (m + n + 1, < \emptyset >) & \text{if } T = R^{\mathbf{o}}; \\ (m + n + 1, < \mathbf{a}(R^{\mathbf{o}}, T) >) & \text{if } T \subset H^{\mathbf{o}} \end{cases}$$

whenever (H, R) is a flag with origin o; $S, T \in \mathbf{R}(o)$;

$$S \subset H; \quad T \subset \mathbf{h}(\mathbf{l}(S), R)^{\mathbf{o}};$$

and $\mathbf{a}(R, S) \in \mathcal{A}$ and $\mathbf{a}(S, T) \subset \mathbf{a}(S, T)$.
Proof. Exercise. □

Theorem. Suppose $(m, \mathcal{A}), (n, \mathcal{B}), (o, \mathcal{C})$ are angle measures. Then

(1) $$(m, \mathcal{A}) + (n, \mathcal{B}) = (n, \mathcal{B}) + (m, \mathcal{A});$$

(2) $$((m, \mathcal{A}) + (n, \mathcal{B})) + (o, \mathcal{C}) = (m, \mathcal{A}) + ((n, \mathcal{B}) + (o, \mathcal{C}));$$

(3) $$(m, \mathcal{A}) + (o, \mathcal{C}) = (n, \mathcal{B}) + (o, \mathcal{C}) \;\Rightarrow\; (m, \mathcal{A}) = (n, \mathcal{B}).$$

(4) $$(m, \mathcal{A}) + (n, \mathcal{B}) \neq (m, \mathcal{A}).$$

Definition. Suppose $(m, \mathcal{A}), (n, \mathcal{B})$ are angle measures. We say

$$(m, \mathcal{A}) < (n, \mathcal{B})$$

if *either* $m < n$ *or* $m = n$, $\mathcal{A} =< \emptyset >$ and $\mathcal{B} \in \mathbf{A}$ *or* $m = n$ and there are angles $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \subset B$ and $A \neq B$.

Theorem. If $(m, \mathcal{A}), (n, \mathcal{B})$ are angle measures then exactly one of the following holds:

$$(m, \mathcal{A}) < (n, \mathcal{B}); \quad (m, \mathcal{A}) = (n, \mathcal{B}); \quad (n, \mathcal{B}) < (m, \mathcal{A}).$$

If $(m, \mathcal{A}), (n, \mathcal{B}), (o, \mathcal{C})$ are angle measures then

$$(m, \mathcal{A}) < (n, \mathcal{B}) \quad \text{and} \quad (n, \mathcal{B}) < (o, \mathcal{C}) \;\Rightarrow\; (m, \mathcal{A}) < (o, \mathcal{C}).$$

If $(m, \mathcal{A}), (n, \mathcal{B})$ are angle measures then

$$(m, \mathcal{A}) < (n, \mathcal{B}) \;\Leftrightarrow\; \text{there is an angle measure } (o, \mathcal{C}) \text{ such that } (m, \mathcal{A}) + (o, \mathcal{C}) = (n, \mathcal{B}).$$

Proof. Exercise using the previous Theorems. □

Definition. We say the line L is a **bisector of the angle** A if the vertex of A lies on L, $L \cap A \neq \emptyset$ and if $\mathbf{a}(R, S) \simeq \mathbf{a}(S, T)$ whenever R, T are the sides of A and $S = L \cap A$.

Definition. Suppose $(m, \mathcal{A})$ is an angle neasure. For each positive integer N we define

$$N(m, \mathcal{A})$$

by requiring that $N(m, \mathcal{A}) = (m, \mathcal{A})$ if $N = 1$ and that $N(m, \mathcal{A}) = (N - 1)(m, \mathcal{A}) + (m, \mathcal{A})$ if $N > 1$.

Theorem. Suppose $(m, \mathcal{A})$ is an angle measure. Then

$$(M + N)(m, \mathcal{A}) = M(m, \mathcal{A}) + N(m, \mathcal{A});$$

$$M(N(m, \mathcal{A})) = (MN)(m, \mathcal{A});$$

$$M(m, \mathcal{A}) = N(m, \mathcal{A}) \;\Rightarrow\; M = N;$$

$$M < N \;\Leftrightarrow\; M(m, \mathcal{A}) < N(m, \mathcal{A}).$$

Proof. Simple exercise making use of induction and the foregoing. □

Theorem. Suppose $(m, \mathcal{A}), (n, \mathcal{B})$ are angle measures and N is a positive integer and

$$N(m, \mathcal{A}) = N(n, \mathcal{B}).$$

Then $(m, \mathcal{A}) = (n, \mathcal{B})$.
Proof. Exercise. □

Definition. Suppose N is a positive integer and $(m, \mathcal{A}$ is an angle measure. Keeping in mind the previous Theorem we let

$$\frac{(m, \mathcal{A})}{N}$$

to be that angle measure such that, if it exists, is such that

$$N\Big(\frac{(m, \mathcal{A})}{N}\Big) = (m, \mathcal{A}).$$

Theorem. Every angle has a unique bisector.
Proof. The existence of bisectors of angles has already been established.

Let R, S be distinct rays with a common origin; let $A = \mathbf{a}(R, S)$; let L and M be bisectors of A and let $U = L \cap A$ and $V = M \cap A$. Let $\mathcal{A} =< \mathbf{a}(R, U) >$ and let $\mathcal{B} =< \mathbf{a}(R, V) >$. Then $2(0, \mathcal{A}) = 2(0, \mathcal{B})$ so, by the previous Theorem, $\mathcal{A} = \mathcal{B}$. That $U = V$ follows from (CA). $\square$

Exterior angle theorem. Suppose T is a triangle with vertices a, b, c; A_a, A_b, A_c are the corresponding angles; $X_c = \mathbf{a}(\mathbf{r}(c, a), \mathbf{r}(c, b)^{\mathbf{o}})$ and $Y_c = \mathbf{a}(\mathbf{r}(c, b), \mathbf{r}(c, a)^{\mathbf{o}})$. Then

$$A_a < X, \quad A_b < X_c, \quad A_b < Y_c, \quad A_b < Y_c.$$

Proof. Let L_a, L_b, L_c be the lines containing the sides of T opposite a, b, c, respectively. Let H_a, H_b, H_c be the halfspaces on the sides of L_a, L_b, L_c containing a, b, c, respectively.

Were it the case that $A_a \simeq X_c$ then L_a and L_c would be parallel by the alternate interior angle theorem; however, b is on both of these lines.

Suppose $X_c < A_a$. Let $R_a = \mathbf{r}(a, c)$. There would be a ray S_a in $\mathbf{A}(H_a, R)$ such that $D_a = \mathbf{a}(R_a, S_a) \simeq X_c$. Since $D_a < A_a$ we have $S_a \subset A_a$. By the Crossbar Theorem S_a meets $\mathbf{s}(b, c)$. Using the alternate interior angle theorem once again we find that $\mathbf{l}(S_a)$ and $\mathbf{l}(b, c)$ would be parallel, which is impossible.

Thus $A_a < X_c$ and $A_a < Y_c$ since Y_c is vertical to X_c. Interchanging a and b in the result just established we infer the other two inequalities. $\square$

Definition. Translations. Suppose a, b are distinct points and $L = \mathbf{l}(a, b)$. We let

$$\tau_{a,b}$$

be that motion which carries $\mathbf{r}(a, b)^{\mathbf{o}}$ to $\mathbf{r}(b, a)^{\mathbf{o}}$ and which preserves the members of $\mathbf{H}(L)$; note that

$$\tau_{a,b}(a) = b.$$

We call $\tau_{a,b}$ **translation from** b **to** a.

Theorem. Suppose a, b are distinct points. Then

$$\tau_{a,b}^{-1} = \tau_{b,a}$$

and

$$\tau_{a,b}(x) \neq x \quad \text{whenever } x \in L.$$

Suppose a, b, c are distinct collinear points. Then

$$\tau_{a,c} = \tau_{b,c} \circ \tau_{a,b}.$$

Proof. This follows directly from uniqueness. $\square$

Remark. As we shall see later, the hypothesis that a, b, c be collinear is not necessary in Euclidean geometry but is necessary in hyperbolic geometry.

Definition. Rotations. Suppose o is a point and R, S are distinct members of $\mathbf{R}(o)$. We let

$$\zeta_{R,S}$$

be the motion which carries R to S and which carries H to I where H is the halfspace on the S side of the line of R and where I is the halfspace opposite the halfspace on the R side of the line of S. We call $\zeta_{R,S}$ the **rotation through the oriented angle determined by** R **and** S**.**

Theorem. Suppose o is a point, R, S, T are distinct members of $\mathbf{R}(o)$ and $S \subset \mathbf{a}(R, T)$. Then

$$\zeta_{R,T} = \zeta_{S,T} \circ \zeta_{R,S}.$$

Proof. Use uniqueness and one of the corollaries of the Crossbar Theorem. □

Theorem. Suppose o is a point, R, S are distinct members of $\mathbf{R}(o)$ and ρ is that motion which carries R to S and which carries H to I where H is the halfspace on the S side of the line of R and where I is the halfspace on the R side of the line of S. Then ρ is a reflection across a line M such that $M \cap H \cap I \neq \emptyset$. In particular, $\rho[S] = R$.
Remark. This Theorem is essential for angle ordering which we carry out below. The proof uses the Continuity Axiom.
Proof. From earlier work there is a ray $T \in \mathbf{A}(R, S)$ such that $\rho[T] = T$. It follows from uniqueness that $\rho^2 = \iota$. By an earlier Theorem, ρ must be either a reflection or the half turn about o. Since $\rho[T] = T$ and not $T^{\mathbf{o}}$ it cannot be the half turn about o. □

The Ruler Theorem. Suppose o and p are distinct points and let $<$ be the geometric linear ordering of $\mathbf{l}(o, p)$ such that $o < p$. There is one and only one order preserving function

$$f_{o,p} : \mathbf{R} \to \mathbf{l}(o, p)$$

such that $f_{o,p}(0) = o$, $f_{o,p}(1) = p$, $f_{o,p}$ is univalent and

$$\mathbf{s}(f_{o,p}(s_1), f_{o,p}(s_2)) \simeq \mathbf{s}(f_{o,p}(t_1), f_{o,p}(t_2)) \;\Leftrightarrow\; |s_1 - s_2| = |t_1 - t_2|$$

whenever s_1, s_2, $s_1 \neq s_2$, $t_1, t_2 \in \mathbf{R}$ and $t_1 \neq t_2$.

Moreover, $\mathbf{rng}\, f_{o,p} = \mathbf{l}(o, p)$.
Proof. We define q_n, $n = 1, 2, 3, \ldots$, inductively as follows. Let $q_1 = p$ and let q_{n+1} be that point of $\mathbf{s}(o, q_n)$ such that $\mathbf{s}(o, q_{n=1}) \simeq \mathbf{s}(q_{n+1}, q_n)$. For each integer m and each positive integer m we let $q_{m,n} = \tau^m_{o,q_n}(o)$. Note that

$$q_{m_1,n_1} = q_{m_2,n_2} \;\Leftrightarrow\; \frac{m_1}{2^{n_1}} = \frac{m_2}{2^{n_2}}. \tag{1}$$

For each positive integer n let $D_n = \{\frac{m}{2^n} : m \text{ is an integer}\}$. Let $D = \cup_{n=1}^{\infty} D_n$. (The members of D are called dyadic rationals.) In view of (1) we may define

$$g : D \to \mathbf{l}(o, p)$$

by requiring that

$$g(\frac{m}{2^n}) = q_{m,n} \text{ whenever } m \text{ is a integer and } n \text{ is a positive integer.}$$

We let

$$f_{o,p}(r) = \sup\{g(d) : d \in D \text{ and } d < r\} \quad \text{whenever } r \in \mathbf{R}.$$

We leave it as a straightforward exercise for the reader making use of previous Theorems to show that $f_{o,p}$ has the desired properties.

Theorem. Length. Suppose o and p are distinct points. There is one and only way to assign to each segment S a **length**

$$|S|$$

such that

$$|\mathbf{s}(o, p)| = 1$$

and such that if S and T are segments then

$$S \simeq T \;\Leftrightarrow\; |S| = |T|$$

and

$$S < T \;\Leftrightarrow\; |S| < |T|.$$

Moreover, if a, c are distinct points and $b \in \mathbf{s}(a, c)$ then

$$|\mathbf{s}(a,c)| = |\mathbf{s}(a,b)| + |\mathbf{s}(b,c)|.$$

Proof. Suppose a, b are distinct points. Let c be that point in $\mathbf{r}(o,p)$ such that $\mathbf{s}(o,c) \simeq \mathbf{s}(a,b)$. Let $|S|$ be such that $f_{o,p}(|S|) = c$. $\square$

Moreover, if (H,R) is a flag and $\theta \in (0,\pi)$ there is one and only member S of $\mathbf{A}(H,R)$ such that $|\mathbf{a}(R,S)| = \theta$; furthermore, if $S, T \in \mathbf{A}(H,R)$ and $S \prec T$ where $\prec$ is the geometric linear order on $\mathbf{A}(H,R)$ then

$$|\mathbf{a}(R,T)| = |\mathbf{a}(R,S)| + |\mathbf{a}(S,T)|.$$

Proof. We proceed as we did in the proof of the Ruler Theorem.

Let (H,R) be a flag and let o be the origin of R. Let S be that ray in $\mathbf{A}(H,R)$ such that $\mathbf{a}(R,S)$ is a right angle. We define the sequence $T_n,\ n = 1,2,3,\ldots,$ as follows. $T_1 = S$ and T_{n+1} is that member of $\mathbf{A}(H,R)$ lying on the bisector of $\mathbf{a}(R,T_n)$. For each ordered pair (m,n) of positive integers such that $m < 2^n$ we let $T_{m,n} = \zeta^m_{R,T_n}(R) \in \mathbf{A}(H,R)$. Note that

(1) $$q_{m_1,n_1} = q_{m_2,n_2} \;\Leftrightarrow\; \frac{m_1}{2^{n_1}} = \frac{m_2}{2^{n_2}}.$$

For each positive integer n let $E_n = \{\frac{m}{2^n} : m \text{ is a positive integer and } m < 2^n\}$. Let $E = \cup_{n=1}^{\infty} E_n$. In view of (1) we may define

$$g : E \to \mathbf{A}(H,R)$$

by requiring that

$$g(\frac{m}{2^n}) = T_{m,n} \text{ whenever } m \text{ and } n \text{ are positive integer and } m < 2^n.$$

Let $\prec$ be the geometric linear ordering of $\mathbf{A}(H,R)$. Let

$$h_{H,R}(\theta) = \sup\{g(e) : e \in E \text{ and } e\pi < \theta\} \quad \text{whenever } \theta \in (0,\pi).$$

We leave to the reader the straightforward exercise of using the preceding theory to show that $h_{H,R}$ is an order preserving mapping carrying $(0,\pi)$ onto $\mathbf{A}(H,R)$ and to prove the remaining assertions of the Theorem.

The book.

Here are the congruence axioms in the book.

There are an equivalence relation $\simeq_S$ on the family of segments and and equivalence relation $\simeq_A$ on the family of angles such that

(BC1) if a and b are distinct points and r is a ray with origin o then there is one and only point p in r such that $\mathbf{s}(a,b) \simeq_S \mathbf{s}(o,p)$.

(BC2) if for each $i = 1,2$, a_i, b_i and c_i are distinct points and $b_i \in \mathbf{s}(a_i,c_i)$; and if $\mathbf{s}(a_1,b_1) \simeq_S \mathbf{s}(a_2,b_2)$ and $\mathbf{s}(b_1,c_1) \simeq_S \mathbf{s}(b_2,c_2)$ then $\mathbf{s}(a_1,c_1) \simeq_S \mathbf{s}(a_2,c_2)$.

(BC3) if A is an angle and (H, r) is a flag there is a unique angle B such that r is a side of B, $B \subset H$ and $A \simeq_A B$.

(BC4) (SAS) Suppose

(i) T_i is a triangle; a_i, b_i, c_i are the vertices of T_i; A_i, B_i, C_i are the angles of T_i corresponding to a_i, b_i, c_i,respectively; and U_i, V_i, W_i are the sides of T_i opposite a_i, b_i, c_i, respectively; and

(ii) $V_1 \simeq_S V_2$, $A_1 \simeq_A A_2$ and $W_1 \simeq_S W_2$.

Then $B_1 \simeq_A B_2$, $U_1 \simeq_S U_2$ and $C_1 \simeq_A C_2$.

Exercise. Show that the incidence axioms, the betweenness axioms and (BC1)-(BC4) imply (Cong).