The Betweenness Axioms.

For each pair of distinct points a and b there is a subset

$$\mathbf{s}(a,b),$$

called the **segment joining** a **to** b, satisfying (B1)-(B3) below. Whenever a and c are distinct points we say the point b is **between** a **and** c if $b \in \mathbf{s}(a,c)$.

(B1) If a and b are distinct points then

(a) $\mathbf{s}(a,b) \subset \mathbf{l}(a,b)$;

(b) $\mathbf{s}(b,a) = \mathbf{s}(a,b)$; and

(c) $\mathbf{s}(b,a) \cap \{a,b\}$ is empty.

(B2)

(a) If a and b are distinct points then $\mathbf{s}(a,b)$ is nonempty.

(b) If a, b and c are distinct and $\{a,b,c\}$ is collinear exactly one of the following holds:

$$a \in \mathbf{s}(b,c), \quad b \in \mathbf{s}(c,a), \quad c \in \mathbf{s}(a,b).$$

(c) If a and b are distinct points there is a point c not equal to a or b such that $b \in \mathbf{s}(a,c)$.

Suppose L is a line and a and b are distinct points not lying on L. We say a **and** b **are on the same side of** L if $\mathbf{s}(a,b)$ does not meet L and we say a **and** b **are on opposite sides of** L if $\mathbf{s}(a,b)$ does meet L. From (B1)(b) we infer that these relations are symmetric in a and b.

The previous axioms imply that a line is at most one dimensional in the sense of topology, whatever that means. The following axiom says that the set of points is at most two dimensional in the sense of topology, whatever that means. The continuity axiom, which will follow, will allow us to replace "at most" with "exactly".

(B3) Suppose L is a line and a,b and c are distinct points not lying on L. Then

(a) if a and b are on the same side of L and b and c are on the same side of L then a and c are on the same side of L.

(b) if a and b are on opposite sides of L and b and c are on opposite sides of L then a and c are on the same side of L.

If S is a segment we say a point a is an **endpoint of** S if there is a point b not equal to a such that $S = \mathbf{s}(a,b)$.

Suppose a and b are distinct points; c and d are distinct points; and $\mathbf{s}(a,b) = \mathbf{s}(c,d)$. It is natural to ask if $\{a,b\} = \{c,d\}$, or, equivalently, if a segment has exactly two endpoints. As we will show below, this is indeed the case.

Theorem. Suppose L is a line and o, a, b and c are distinct points lying on L. Then

(B4)(c) If $o \notin \mathbf{s}(a,b)$ and $o \notin \mathbf{s}(b,c)$ then $o \notin \mathbf{s}(a,c)$.

(B4)(d) If $o \in \mathbf{s}(a,b)$ and $o \in \mathbf{s}(b,c)$ then $o \notin \mathbf{s}(a,c)$.

Proof. Let d be a point not lying on L; we have show that such a point exists as a consequence of the Incidence Axioms. Let M be the line determined by o and d.
Lemma. Suppose X is a subset of L. Then $X \cap M = X \cap \{o\}$.
Proof. Since $o \in M$ we have $X \cap \{o\} \subset X \cap M$. Suppose $x \in X \cap M$. Were $x \neq o$ we the two member set $\{x, o\}$ would be a subset of each of the distinct lines lines L and M which contradicts (I1). □

Now apply (B3) with L there replaced by M. □

The following Theorem is a fundamental tool in dealing with betweenness.
The Splitting Theorem. Suppose a and c are distinct points and b is between a and c. Then $b \notin \{a, c\}$ and

$$\{\mathbf{s}(a,b), \{b\}, \mathbf{s}(b,c)\} \text{ is a partition of } \mathbf{s}(a,c).$$

Remark. Although it is a bit tedious, the reader should check this proof very carefully. The writer claims he has put in *all* the steps!
Proof. That $b \notin \{a, c\}$ follows from (B1)(c). Suppose d is a point on the line determined by a and c.

Suppose $d \in \mathbf{s}(a,b)$. Then $d \notin \{a, b\}$ by (B1)(b). Since $c \notin \mathbf{s}(a,b)$ by (B2)(b) we have $d \neq c$. Thus the points a, b, c, d are distinct by (B1)(c). From (B2)(b) we infer that $b \notin \mathbf{s}(a,d)$. Were it the case that $b \notin \mathbf{s}(c,d)$ (B4)(c) would imply $b \notin \mathbf{s}(a,c)$ so $b \in \mathbf{s}(c,d)$. From (B2)(b) it follows that $d \notin \mathbf{s}(b,c)$. Were $d \notin \mathbf{s}(c,a)$ we would have $d \notin \mathbf{s}(b,a)$ by (B4)(c) which in turn would imply by (B1)(b) that $d \notin \mathbf{s}(a,b)$; thus $d \in \mathbf{s}(c,a)$. By (B1)(b), $d \in \mathbf{s}(a,c)$. Thus $\mathbf{s}(a,b) \subset \mathbf{s}(a,c)$. Interchanging a and c and using (B1)(b) twice we infer that $\mathbf{s}(b,c) \subset \mathbf{s}(a,c)$. Thus

$$\mathbf{s}(a,b) \cup \mathbf{s}(b,c) \subset \mathbf{s}(a,c). \qquad (1)$$

If d is a point on $\mathbf{l}(a,c)$, $d \notin \{a,c\}$, $d \notin \mathbf{s}(a,b)$ and $d \notin \{a,b,c\}$ then, by (B4)(c), $d \notin \mathbf{s}(a,c)$. Thus

$$\mathbf{s}(a,c) \subset \mathbf{s}(a,b) \cup \{b\} \cup \mathbf{s}(b,c).$$

If $d \in \mathbf{s}(a,b) \cap \mathbf{s}(b,c)$ then $d \notin \mathbf{s}(a,c)$ by (B4)(c); this is incompatible with (1). That $\{\mathbf{s}(a,b), \{b\}, \mathbf{s}(b,c)\}$ is disjointed now follows from (B1)(c) □
Remark. Having gotten through that we will now start being informal. Watch out! Many good mathematicians have gotten egg on their face in this subject.

The following notion will shortly prove to be convenient.
Definition. Suppose X is a set of points. We let

$$\mathbf{b}(X),$$

the **boundary of** X, be the set of points b such that there is a point $a \in X$ such that $\mathbf{s}(a,b) \subset X$. Note that if X is convex in the standard Euclidean model then $\mathbf{b}(X)$ is the boundary of X with respect to the standard topology on $\mathbf{R}^2$.

Definition. We say a set C of points is **convex** if

$$\mathbf{s}(a,b) \subset C \text{ whenever } a, b \in C.$$

Theorem. Suppose $\mathcal{C}$ is a nonempty family of convex sets of points. Then $\cap\mathcal{C}$ is convex.
Proof. This is obvious. □

Rays.

Definition. Suppose o and a are distinct points. We let

$$\mathbf{r}(o,a) = \{a\} \cup \{b \in \mathbf{l}(o,a) \sim \{a\} : o \notin \mathbf{s}(a,b)\}$$

and we call this set the **ray emanating from** o **passing through** a. For any point o we let

$$\mathbf{R}(o) = \{\mathbf{r}(o,a) : a \text{ is a point and } a \neq o\}.$$

Theorem. Suppose o is a point. Then $\mathbf{R}(o)$ is a partition of the set of points not equal to o.
Proof. It follows directly from (B1)(b) and (B4)(c) that

$$\{(a,a) : a \text{ is a point and } a \neq o\} \cup \{(a,b) : a \text{ and } b \text{ are points, } a \neq o,\ b \neq o,\ a \neq b \text{ and } o \notin \mathbf{s}(a,b)\}$$

is an equivalence relation on the set of points not equal to o. It is immediate that the equivalence classes are the rays $\mathbf{r}(o,a)$ corresponding to points a not equal o. $\square$

Theorem. Suppose o and a are distinct points and L is a line. Then

$$\mathbf{r}(o,a) \subset L \iff L = \mathbf{l}(o,a).$$

Proof. It is an immediate consequence of the definition that $\mathbf{r}(o,a) \subset \mathbf{l}(o,a)$.

By (B2)(a) there is a point b between o and a. By (B1)(c), $a \neq b$. By (B2)(b), $o \notin \mathbf{s}(a,b)$. Thus $\mathbf{r}(o,a)$ contains at least two points and therefore can be contained in at most one line. $\square$

Theorem. Suppose o and a are distinct points. Then $\mathbf{b}(\mathbf{r}(o,a)) = \{o\}$.
Proof. We have $o \notin \mathbf{r}(o,a)$. Suppose $b \in \mathbf{s}(o,a)$. Then $o \notin \mathbf{s}(a,b)$ by (B2)(b) so $b \in \mathbf{r}(o,a)$. That is, $\mathbf{s}(o,a) \subset \mathbf{b}(\mathbf{r}(o,a))$. Thus $o \in \mathbf{b}(\mathbf{r}(o,a))$.

Suppose $p \in \mathbf{b}(\mathbf{r}(o,a))$. Then $p \notin \mathbf{r}(o,a)$ and there is b such that $b \in \mathbf{r}(o,a)$ and

$$\mathbf{s}(p,b) \subset \mathbf{r}(o,a). \tag{1}$$

Suppose $p \neq o$. Then $o \in \mathbf{s}(p,a)$. Since $o \notin \mathbf{s}(a,b)$ we infer from (B4)(c) and (B1)(b) that $o \in \mathbf{s}(p,b)$ which by (1) implies $o \in \mathbf{r}(o,a)$ which is impossible. $\square$

Definition. Suppose R is a ray. In view of the two preceding Theorems, we may define

$$\mathbf{l}(R)$$

as the unique line containing R and we may define

$$\mathbf{o}(R)$$

as the unique member of $\mathbf{b}(R)$. We call $\mathbf{o}(R)$ the **origin of** R.

Definition. Suppose R is a ray. We set

$$R^{\mathbf{o}} = \mathbf{l}(R) \sim (R \cup \{\mathbf{o}(R)\}).$$

Theorem. Suppose R is a ray. Then $R^{\mathbf{o}}$ is ray, $\mathbf{o}(R^{\mathbf{o}}) = \mathbf{o}(R)$ and $\mathbf{l}(R^{\mathbf{o}}) = \mathbf{l}(R)$.
Proof. Suppose o, a are such that $R = \mathbf{r}(o,a)$. Let $L = \mathbf{l}(o,a)$ and let $S = L \sim (\{o\} \cup R)$. By (B2)(c) there is $b \in \mathbf{l}(o,a)$ such that $o \in \mathbf{s}(a,b)$. I claim that $S = \mathbf{r}(o,b)$. Suppose $c \in S$. Then $c \neq o$ and $o \in \mathbf{s}(a,c)$. By (B1)(b) and (B4)(d), $o \notin \mathbf{s}(b,c)$ so $c \in \mathbf{r}(o,b)$. Suppose $d \in \mathbf{r}(o,b)$. Then $d \neq o$ and $o \notin \mathbf{s}(b,d)$. Since $o \in \mathbf{s}(a,b)$ we infer from (B4)(c) that $o \in \mathbf{s}(a,d)$ so $b \notin R$. $\square$

Definition. Suppose R is a ray. We call $R^{\mathbf{o}}$ the **ray opposite** R**.** We have shown that $\{R, \mathbf{o}(R), R^{\mathbf{o}}\}$ is a partition of $\mathbf{l}(R)$.

Definition. Geometric linear orderings. Suppose L is a line and $S \subset L$. We say a linear ordering $<$ of S is **geometric** if

$$S \cap \mathbf{s}(a,b) = \{x \in S : a < x < b\} \quad \text{whenever } a, b \in S \text{ and } a < b.$$

Note that the inverse of a geometric linear ordering of S is a geometric linear ordering of S.

Theorem. Suppose L is a line, $S \subset L$ and $<$ and $\prec$ are geometric linear orderings of S. Then $\prec$ equals $<$ or its inverse.
Proof. Replacing $\prec$ by its inverse if necessary we may there are distinct points a, b of S such that $a < b$ and $a \prec b$. Suppose $x \in S$. Then

$$\begin{aligned} x < a \;&\Leftrightarrow\; a \in \mathbf{s}(x,b) \;\Leftrightarrow\; x \prec a; \\ a < x < b \;&\Leftrightarrow\; x \in \mathbf{s}(a,b) \;\Leftrightarrow\; a \prec x \prec b; \\ b < x \;&\Leftrightarrow\; b \in \mathbf{s}(a,x) \;\Leftrightarrow\; b \prec x; \end{aligned}$$

thus

$$(1) \qquad x < a \;\Leftrightarrow\; x \prec a.$$

Suppose $x, y \in S$. Using (1) repeatedly we find that

$$\begin{aligned} x < y < a \;&\Leftrightarrow\; y \in \mathbf{s}(x,a) \;\Leftrightarrow\; x \prec y \prec a; \\ x < a < y \;&\Leftrightarrow\; a \in \mathbf{s}(x,y) \;\Leftrightarrow\; x \prec a \prec y; \\ a < x < y \;&\Leftrightarrow\; x \in \mathbf{s}(a,y) \;\Leftrightarrow\; a \prec x \prec y. \end{aligned}$$

It follows that $\prec$ equals $<$. □

Lemma. Suppose R is ray with origin o and $<$ is relation on R such that

$$\mathbf{s}(o,b) = \{x \in R : x < b\} \quad \text{whenever } b \in R.$$

Then $<$ is geometric linear ordering $<$ of R.
Proof. Suppose $a, b \in R$ and $a \neq b$. Then $a < b$ if and only if $a \in \mathbf{s}(o,b)$ and $b < a$ if and only if $b \in \mathbf{s}(o,a)$. Thus $<$ is trichotomous.

Suppose $a, b, c \in R$, $a < b$ and $b < c$. Then $a \in \mathbf{s}(o,b)$ and $b \in \mathbf{s}(o,c)$ so the Splitting Theorem implies that $a \in \mathbf{s}(o,c)$ so $<$ is transitive. □

Theorem. Suppose L is a line. There is a geometric linear ordering of L.
Proof. Let $o \in L$ and let R be a ray with origin o. Let $<$ be the union of

$$\{(a,b) \in R^{\mathbf{o}} \times R^{\mathbf{o}} : b \in \mathbf{s}(a,o)\}, \quad R^{\mathbf{o}} \times \{o\}, \quad R^{\mathbf{o}} \times R, \quad \{o\} \times R, \quad \{(a,b) \in R \times R : a \in \mathbf{s}(o,b)\}.$$

Keeping in mind the previous Lemma we find that the restriction of $<$ to $R^{\mathbf{o}} \times R^{\mathbf{o}}$ is a geometric linear ordering of $R^{\mathbf{o}}$ and that the restriction of $<$ to $R \times R$ is a geometric linear ordering of R. It is a simple matter to verify that $<$ is a linear ordering of L.

Suppose $a, b \in L$ and $a < b$. We need to show that

$$(1) \qquad \mathbf{s}(a,b) = \{x \in L : a < x < b\}.$$

In case $(a,b) \in R^{\mathbf{o}} \times R^{\mathbf{o}}$ (1) holds because the restriction of $<$ to $R^{\mathbf{o}} \times R^{\mathbf{o}}$ is a geometric linear ordering of $R^{\mathbf{o}}$.

In case $(a,b) \in R^{\mathbf{o}} \times \{o\}$ (1) holds by the definition of $<$.

In case $(a,b) \in R^{\mathbf{o}} \times R$ we use the Splitting Theorem and the definition of $<$ to infer that

$$\mathbf{s}(a,b) = \mathbf{s}(a,o) \cup \{o\} \cup \mathbf{s}(o,b) = \{x \in L : a < x < o\} \cup \{o\} \cup \{x \in L : o < x < b\} = \{x \in L : a < x < b\}.$$

In case $(a,b) \in \{o\} \times R$ (1) holds by the definition of $<$.

In case $(a,b) \in R \times R$ (1) holds because the restriction of $<$ to $R \times R$ is a geometric linear ordering of R. $\square$

Theorem. Suppose L is a line, $<$ is a geometric linear ordering of L, $o \in L$ and $R \subset L$. Then R is a ray with origin o if and only if either $R = \{x \in L : o < x\}$ or $R = \{x \in L : x < o\}$.

Proof. Simple exercise for the reader. $\square$

Definition. Suppose L is a line, $<$ is a geometric linear ordering of L, R is a ray with origin o and $R \subset L$. We say $<$ is the **geometric linear ordering (of L) induced by** R if $R = \{x \in L : o < x\}$. As a consequence of the previous Theorem we find that there is exactly one geometric ordering of a given line induced by a ray contained in that line.

Theorem. Suppose L is a line; R is a ray with origin o; $R \subset L$; $<$ is the geometric linear ordering of L induced by R; and $p \in R$. Then $\{\{x \in S : o < x < p\}, \{p\}, \{x \in L : p < x\}$ is a partition of L

Proof. Simple exercise for the reader. $\square$

Theorem. Suppose L is a line; R is a ray with origin L and $R \subset L$; S is a ray with origin p and $S \subset L$; and $R \neq S$. Then exactly one of the following holds:

(i) $R \subset S$; (ii) $S \subset R$; (iii) $R \cap S = \emptyset$; (i) $R \cap S$ is a segment.

Furthermore, (i) or (ii) hold if and only if the geometric ordering of L induced by R equals the geometric linear ordering of L induced by S and (iii) or (iv) hold if and only if the geometric ordering of L induced by R equals the inverse of the geometric linear ordering of L induced by S.

Moreover (i) holds if and only if $p \in R$; (ii) holds if and only if $o \in R$; (iii) holds if and only if either $o = p$ and $S = R^{\mathbf{o}}$ or $o \neq p$ and $L \sim (R \cup S \cup \{o,p\}) = \mathbf{s}(o,p)$; and (iv) holds if and only if $o \neq p$ and $R \cap S = \mathbf{s}(o,p)$.

Proof. Simple exercise for the reader. $\square$

The Continuity Axiom. There are reasons not to like this.

(Cont) Any geometric linear ordering of a line is complete.

Halfspaces.

A lot of what we will do here will bear a resemblance to what we did with rays.

Definition. Suppose L is a line and a is a point not lying on L. For each point a not lying on L we let

$$\mathbf{h}(L,a)$$

be the set of those points b such that *either* $b = a$ *or* $b \neq a$, $b \notin L$ and $\mathbf{s}(a,b)$ does not meet L. Thus $\mathbf{h}(L,a)$ is the set of points consisting of a itself and the points which are on the same side of L as a. We set

$$\mathbf{H}(L) = \{\mathbf{h}(L,a) : a \text{ is a point and } a \notin L\}.$$

We say H **is a halfspace** if $H \in \mathbf{H}(L)$ for some line L.

Theorem. Suppose L is a line. Then $\mathbf{H}(L)$ has exactly two members and is a partition of the set of points not lying on L.

Proof. It follows directly from (B) that

$$\{(a,a) : a \text{ is a point and } a \notin L\} \cup \{(a,b) : a \text{ and } b \text{ are points, } a,b \notin L,\ a \neq b \text{ and } \mathbf{s}(a,b) \text{ does not meet } L\}$$

is an equivalence relation on the set of points not lying on L which has at most two equivalence classes.

Let a be a point not lying on L. such a point exists by (I). Let o be a point in L; such a point exists by (I). Let b be a point not equal to o or a such that o is in the segment joining a to b; such a point exists. By (B), a and b are not equivalent. Thus $\mathbf{H}(L)$ has at least two equivalence classes. □

Theorem. Suppose L is a line, a and b are distinct points, $a \notin L$ and $\mathbf{s}(a,b)$ does not meet L. Then

$$\mathbf{s}(a,b) \subset \mathbf{h}(L,a).$$

Proof. Suppose $c \in \mathbf{s}(a,b)$. Then $c \notin L$ and, by the Splitting Theorem, $\mathbf{s}(a,c) \subset \mathbf{s}(a,b)$ so $\mathbf{s}(a,c)$ does not meet L. Thus $c \in \mathbf{h}(L,a)$. □

Theorem. Suppose H is a halfspace. Then $\mathbf{b}(H)$ is a line and

$$H \in \mathbf{H}(\mathbf{b}(H)).$$

Proof. Let L be a line be such that $H \in \mathbf{H}(L)$. Let H' be such that $\mathbf{H}(L) = \{H, H'\}$.

Suppose $b \in \mathbf{b}(H)$. Let $a \in H$ be such that $\mathbf{s}(a,b) \subset H$. Were it the case that $b \notin L$ we would have $b \in H'$ so $\mathbf{s}(a,b)$ would meet L which is impossible. Thus $b \in L$.

Suppose $b \in L$. Then $b \notin H$. Let $a \in H$. Since $a \notin L$ we have that $\mathbf{s}(a,b)$ does not meet L. By the previous Theorem, $\mathbf{s}(a,b) \subset H$ so $b \in \mathbf{b}(H)$. □

Definition. Suppose H is a halfspace. In view of the preceding two Theorems we may define the halfspace

$$H^{\mathbf{o}}$$

by requiring that $\mathbf{H}(\mathbf{b}(H)) = \{H, H^{\mathbf{o}}\}$. We call $H^{\mathbf{o}}$ the **halfspace opposite** H. Evidently,

$$H^{\mathbf{oo}} = H.$$

Theorem. Suppose L is a line, $H \in \mathbf{H}(L)$, $o \in L$ and R is a ray with origin o. Then either $R \subset L$, $R \subset H$ or $R \subset H^{\mathbf{o}}$.

Proof. Let a be such that $R = \mathbf{r}(o,a)$ and suppose $b \in R \sim \{a\}$. If $a \in L$ then $\mathbf{l}(a,b) = \mathbf{l}(o,a) = L$ so $b \in L$. If $a \in H$ then $\mathbf{s}(a,b)$ does not meet L since $\mathbf{l}(a,b)$ can meet L in at most one point so $b \in H$. Similarly, if $a \in H^{\mathbf{o}}$ one shows that $R \subset H^{\mathbf{o}}$. □

Definition. Suppose L is a line, o is a point lying on L, R is a ray with vertex o and

$$L \cap R = \emptyset.$$

Keeping in mind the previous Theorem, we let

$$\mathbf{h}(L,R)$$

be that member H of $\mathbf{H}(L)$ such that $R \subset H$.

Theorem. Suppose H is a halfspace, C is a convex set of points and $C \cap \mathbf{b}(H) = \emptyset$. Then either $C \subset H$ or $C \subset H^{\mathbf{o}}$.

Proof. Straightforward exercise for the reader. □

Theorem. Lines, segments and rays and halfspaces are convex.

Proof. Exercise for the reader. $\square$

Theorem. Suppose L is a line and H is a halfspace. Then exactly one of the following holds:

(i) $L \subset H$;

(ii) $L \subset H^{\mathbf{o}}$;

(iii) $L = \mathbf{b}(H)$;

(iv) there is a point o such that $L \cap \mathbf{b}(H) = \{o\}$ and $L \cap H$ is a ray with origin o.

Proof. Since L is convex, if L does not meet $\mathbf{b}(H)$ then either (i) or (ii) holds.

If L meets $\mathbf{b}(H)$ in more than one point then (iii) holds.

Suppose o is a point such that $L \cap \mathbf{b}(H) = \{o\}$. Use (B2)(b) to choose a point a in $L \cap H$. We leave it as a simple exercise for the reader to verify that $L \cap H = \mathbf{r}(o, a)$. $\square$

Fundamental sets.

We now study intersections of finite families of halfspaces.

Definition. We say a set of points F is **fundamental** if F is nonempty and

$$F = \cap \mathcal{F}$$

for some finite nonempty family $\mathcal{F}$ of halfspaces.

Theorem. Suppose F is a fundamental set of points, $a \in \mathbf{b}(F)$ and $b \in F$. Then $\mathbf{s}(a, b) \subset F$.
Proof. Let $\mathcal{F}$ be family of halfspaces such that $F = \cap \mathcal{F}$. Since $a \in \mathbf{b}(F)$ there is $c \in F$ such that $\mathbf{s}(a, c) \subset F$. Let $H \in \mathcal{F}$. Then $\mathbf{s}(a, c) \subset H$ so either $a \in \mathbf{b}(H) \cup H$. This implies $\mathbf{s}(a, b) \subset H$. $\square$

Definition. We say a family $\mathcal{E}$ of halfspaces is **efficient** if $\mathcal{E}$ is finite and nonempty, $\cap \mathcal{E} \neq \emptyset$ and $\cap \mathcal{F} \neq \cap \mathcal{E}$ whenever $\mathcal{F}$ is a nonempty proper subfamily of $\mathcal{E}$.

Theorem. Suppose $\mathcal{F}$ is a nonempty family of halfspaces and $\cap \mathcal{F} \neq \emptyset$. Then there is an efficient subfamily $\mathcal{E}$ of $\mathcal{F}$ such that $\cap \mathcal{E} = \cap \mathcal{F}$.
Proof. Let $\mathbf{G}$ be the class of nonempty subfamilies $\mathcal{G}$ of $\mathcal{F}$ such that $\cap \mathcal{F} = \cap \mathcal{G}$. Note that $\mathcal{F} \in \mathbf{G}$ so $\mathbf{G}$ is nonempty and therefore

$$m = \min\{\mathbf{card}\, \mathcal{G} : \mathcal{G} \in \mathbf{G}\}$$

is a well defined positive integer. Any member $\mathcal{E}$ of $\mathbf{G}$ such that $\mathbf{card}\, \mathcal{E} = m$ is efficient. $\square$

Remark. We will show below that such an $\mathcal{E}_i,\ i = 1, 2$, are efficient families of halfspaces and $\cap \mathcal{E}_1 = \cap \mathcal{E}_2$ then $\mathcal{E}_1 = \mathcal{E}_2$.

Corollary. Suppose F is a fundamental set of points. Then there is an efficient family $\mathcal{E}$ of halfspaces such that $\mathcal{E} \subset \mathcal{F}$ and $F = \cap \mathcal{E}$.

Theorem. Suppose H and I are distinct halfspaces. Then exactly one of the following holds:

(i) $\mathbf{b}(H)$ meets $\mathbf{b}(I)$ in a single point and $\{H, I\}$ is efficient.

(ii) $\mathbf{b}(H) = \mathbf{b}(I)$, $I = H^{\mathbf{o}}$, $H \cap I$ is empty, and $\{H, I\}$ is not efficient.

(iii) $\mathbf{b}(I) \subset H$, $\mathbf{b}(H) \subset I$ and $\{H, I\}$ is efficient.

(iv) $\mathbf{b}(I) \subset H$, $\mathbf{b}(H) \subset I^{\mathbf{o}}$, $H \subset I$, and $\{H, I\}$ is not efficient.

(v) $\mathbf{b}(I) \subset H^{\mathbf{o}}$, $\mathbf{b}(H) \subset I$, $I \subset H$, and $\{H, I\}$ is not efficient.

(vi) $\mathbf{b}(I) \subset H^{\mathbf{o}}$, $\mathbf{b}(H) \subset I^{\mathbf{o}}$, $H \cap I$ is empty, and $\{H, I\}$ is not efficient.

Remark. Draw a picture.
Remark. (i) implies that $H \cap I$ is not empty whenever H and I are halfspaces such that $\mathbf{b}(H)$ and $\mathbf{b}(I)$ meet in a single point.
Proof. Suppose o is a point such that $\mathbf{b}(H) \cap \mathbf{b}(I) = \{o\}$. Choose a point a in $\mathbf{b}(H) \cap I$ and a point b in $\mathbf{b}(I) \cap H$; such points exist by a previous Theorem. Then $\mathbf{s}(a, b) \subset H \cap I$ by a previous Theorem so $H \cap I$ is nonempty. By (B) there are points c, d not equal to a or b such that $b \in \mathbf{s}(a, c)$ and $a \in \mathbf{s}(b, d)$. We have $c \in H \sim I$ and $d \in I \sim H$ so $H \cap I$ is not equal to either H or I. Thus $\{H, I\}$ is efficient and (i) holds.

In case $\mathbf{b}(H)$ meets $\mathbf{b}(I)$ in more that one point we have $\mathbf{b}(H) = \mathbf{b}(I)$ so $I = H^{\mathbf{o}}$ and (ii) holds.

Suppose $\mathbf{b}(H)$ does not meet $\mathbf{b}(I)$. Keeping in mind that lines are convex we find that *either* (vii) $\mathbf{b}(I) \subset H$ and $\mathbf{b}(H) \subset I$; *or* (viii) $\mathbf{b}(I) \subset H$ and $\mathbf{b}(I) \subset H^{\mathbf{o}}$; *or* (ix) $\mathbf{b}(I) \subset H^{\mathbf{o}}$ and $\mathbf{b}(I) \subset H$; *or* (x) $\mathbf{b}(I) \subset H^{\mathbf{o}}$ and $\mathbf{b}(I) \subset H^{\mathbf{o}}$. If (vii) holds we proceed as in the first paragraph to show that $\{H, I\}$ is efficient. We leave it as an exercise for the reader to show that (iv), (v) and (vi) hold if (viii), (ix) or (x) hold, respectivley. □

The Crossbar Theorem. Suppose o is a point and H and I are halfspace such that $\mathbf{b}(H) \cap \mathbf{b}(I) = \{o\}$. Suppose $a \in \mathbf{b}(H) \cap I$ and $b \in \mathbf{b}(I) \cap H$. Then

$$H \cap I = \cup\{\mathbf{r}(o, e) : e \in \mathbf{s}(a, b)\}. \tag{1}$$

Remark. a and b exist by virtue of a previous Theorem.
Remark. As we shall see, in hyperbolic geometry it is *not* the case that if $d \in H \cap I$ then there is a segment containing d joining a point of $\mathbf{b}(H) \cap I$ to a point of $\mathbf{b}(I) \cap H$.
Proof. Let $L = \mathbf{b}(H)$ and let $M = \mathbf{b}(I)$. Suppose $e \in \mathbf{s}(a, b)$. Note that $\mathbf{s}(a, b)$ does not meet either L or M. By the Splitting Theorem, $\mathbf{s}(a, e)$ and $\mathbf{s}(b, e)$ are subsets of $\mathbf{s}(a, b)$. Since $\mathbf{s}(b, e)$ does not meet L it follows that $e \in \mathbf{h}(L, b) = H$ and since $\mathbf{s}(a, e)$ does not meet M it follows that $e \in \mathbf{h}(M, a) = I$. Thus $e \in H \cap I$. Since $e \notin L \cup M$ we have that $\mathbf{r}(o, e)$ does meet L or M. If $x \in \mathbf{r}(o, e)$ then $\mathbf{s}(x, e)$ does not meet either L or M because $\mathbf{s}(x, e) \subset \mathbf{r}(o, e)$. Thus $x \in H \cap I$. Thus the right hand side of (1) is a subset of $H \cap I$.

Suppose $d \in H \cap I$. Let $N = \mathbf{l}(o, d)$ and note that N does not meet $H \cap I^{\mathbf{o}}$ because were $x \in N \cap H \cap I^{\mathbf{o}}$ we would have $o \in \mathbf{s}(x, d)$ because $d \in I$ and $o \notin \mathbf{s}(x, d)$ because $d \in H$. Choose $c \in L \cap I^{\mathbf{o}}$. Since $\mathbf{s}(b, c)$ meets neither L nor M it is a subset of $H \cap I^{\mathbf{o}}$. Thus $\mathbf{s}(b, c)$ does not meet N. But $\mathbf{s}(a, c)$ meets N in o so (B) implies $\mathbf{s}(a, b)$ meets N in a point e. Since $\mathbf{s}(d, e) \subset H \cap I$ we find that $d \in \mathbf{r}(o, e)$. Thus $H \cap I$ is a subset of the right hand side of (1). □

Corollary. Suppose H and I are distinct halfspaces; N is a line; o is a point such that $\mathbf{b}(H) \cap \mathbf{b}(I) \cap N = \{o\}$; and

$$J = \mathbf{h}(N, \mathbf{b}(H) \cap I).$$

Then *either* $H \cap I \subset J$ *or* $\{H \cap J, I \cap J^{\mathbf{o}}\}$ is a partition of $(H \cap I) \sim N$.
Proof. Suppose $H \cap I \cap N = \emptyset$. Then, as $H \cap I$ is convex and nonempty, there is a unique halfspace J such that $\mathbf{b}(J) = N$ and $H \cap I \subset J$.

Suppose $H \cap I \cap N \neq \emptyset$. Let a, b be such that $a \in I \cap \mathbf{b}(J)$, $b \in H \cap \mathbf{b}(I)$. By the Crossbar Theorem,

$$H \cap I = \cup\{\mathbf{r}(o, c) : c \in \mathbf{s}(a, b)\}.$$

$H \cap N$ and $I \cap N$ are rays with origin o which are subsets of N and have a point in common so they both equal $H \cap I \cap N$. Thus there is a unique member c of $\mathbf{s}(a, b)$ such that $\mathbf{r}(o, c) = H \cap I \cap N$. Let $J = \mathbf{h}(N, a)$ and note that $J^{\mathbf{o}} = \mathbf{h}(N, b)$. Using the Crossbar Theorem twice we find that

$$H \cap J = \cup\{\mathbf{r}(o, d) : d \in \mathbf{s}(a, c)\} \quad \text{and} \quad I \cap J^{\mathbf{o}} = \cup\{\mathbf{r}(o, e) : d \in \mathbf{s}(c, b)\}.$$

Thus the Theorem is proved.

Theorem. Suppose H, I and J are distinct halfspace and $\mathbf{b}(H)$, $\mathbf{b}(I)$ and $\mathbf{b}(J)$ meet in the point o. Then $\{H, I, J\}$ is not efficient.
Proof. Applying the previous Theorem with $N = \mathbf{b}(J)$ we find that *either* (i) $H \cap I \subset J$, *or* (ii) $H \cap I \subset J^{\mathbf{o}}$, *or* (iii) $\{H \cap J, I \cap J^{\mathbf{o}}\}$ is a partition of $H \cap I \sim \mathbf{b}(J)$ or (iv) $\{H \cap J^{\mathbf{o}}, I \cap J\}$ is a partition of $H \cap I \sim \mathbf{b}(J^{\mathbf{o}})$. In case (i) holds we have $H \cap I = H \cap I \cap J$. In case (ii) holds we have $H \cap I \cap J = \emptyset$. In case (iii) holds we have $H \cap I \cap J = H \cap J$. In case (iv) holds we have $H \cap I \cap J = I \cap J$.

Corollary. Suppose $\mathcal{E}$ is efficient family of halfspaces and o is a point. Then

$$\{H \in \mathcal{E} : o \in \mathbf{b}(H)\}$$

contains at most two members.

Theorem. Suppose L is a line, $\mathcal{F}$ is a finite family of halfspaces,

$$S = \cap\{L \cap \mathbf{b}(H) : H \in \mathcal{F}\}$$

and $S \neq \emptyset$.

Then S is a segment, a ray or the line itself. Furthermore,

$$\mathbf{b}(S) \subset \cup\{L \cap \mathbf{b}(H) : H \in \mathcal{F}\}.$$

Proof. Let $\mathcal{G}$ be the set of those H in $\mathcal{F}$ such $L \cap \mathbf{b}(H)$ is a ray with origin in L and let $\mathcal{O}$ be the set of origins of these rays. By a previous Theorem, $L \cap \mathbf{b}(H) = L$ if $H \in \mathcal{F} \sim \mathcal{G}$. If $\mathcal{G}$ is empty then $S = L$ and the Theorem holds, so let us assume $\mathcal{G}$ is not empty.

Let $<$ be a geometric linear ordering of L. Let $\mathcal{O}^+$ be the set of $o \in \mathcal{O}$ such that there is $H \in \mathcal{G}$ such that $L \cap \mathbf{b}(H) = \{x \in L : o < x\}$ and let $\mathcal{O}^-$ be the set of $o \in \mathcal{O}$ such that there is $H \in \mathcal{G}$ such that $L \cap \mathbf{b}(H) = \{x \in L : x < o\}$. Evidently, $\mathcal{O}$ is the disjoint union of $\mathcal{O}^-$ and $\mathcal{O}^+$.

In case $\mathcal{O}^-$ is empty, we let b be the $<$-largest member of $\mathcal{O}^+$ and note that $S = \{x \in L : b < x\}$.

In case $\mathcal{O}^+$ is empty, we let a be the $<$-least member of $\mathcal{O}^-$ and note that $S = \{x \in L : x < a\}$.

In case neither $\mathcal{O}^-$ nor $\mathcal{O}^+$ is empty we let a be the $<$-largest member of $\mathcal{O}^-$, we let b be the $<$-largest member of $\mathcal{O}^+$ and note that $S = \{x \in L : a < x < b\}$. $\square$

Definition. Suppose $\mathcal{F}$ is a nonempty family of halfspaces. For any H in $\mathcal{F}$ we let

$$\mathbf{s}(\mathcal{F}, H) = \begin{cases} \mathbf{b}(H) & \text{if } \mathcal{F} = \{H\}, \\ \cap\{\mathbf{b}(H) \cap G : G \in \mathcal{F} \sim \{H\}\} & \text{else.} \end{cases}$$

Theorem. Suppose $\mathcal{F}$ is an nonempty family of halfspaces, $F = \cap\mathcal{F}$ and F is nonempty.

If $H \in \mathcal{F}$ and $S = \mathbf{s}(\mathcal{F}, H)$ then S is either empty or is a line, a ray or a segment; moreover, if S is empty then $F = \cup(\mathcal{F} \sim \{H\})$.

Furthermore,

$$\{\mathbf{s}(\mathcal{F}, H) : H \in \mathcal{F}\} \cup \{\cup\{\mathbf{b}(\mathbf{s}(\mathcal{F}, H)) : H \in \mathcal{E}\}\}$$

is a partition of $\mathbf{b}(\cap\mathcal{F})$.
Proof. The Theorem holds trivially if $\mathcal{F}$ has exactly one member, so let us suppose $\mathcal{F}$ has at least two members.

Suppose $H \in \mathcal{F}$ and let $S = \mathbf{s}(\mathcal{F}, H)$.

Suppose S is empty. Let $\mathcal{G} = \mathcal{F} \sim \{H\}$. Then $\cap\mathcal{G}$ is a convex set which does not meet $\mathbf{b}(H)$ so is either a subset of H or $H^{\mathbf{o}}$. Were it a subset of $H^{\mathbf{o}}$ we would have $F = H \cap (\cap\mathcal{G}) \subset H \cap H^{\mathbf{o}} = \emptyset$ which is impossible. Thus $F = H \cap (\cap\mathcal{G}) = \cap\mathcal{G}$.

Now suppose S is nonempty. That S is a line, a ray or a segment follows from the previous Theorem.

Choose a point c in F.

Suppose $b \in S$. Then $b \in \mathbf{b}(H)$ so $b \notin P$. By a previous Theorem, $\mathbf{s}(b,c) \subset H$. Suppose $G \in \mathcal{F} \sim \{H\}$. Since G is a halfspace and $b \in G$ we have $\mathbf{s}(b,c) \subset G$. Thus $b \in \mathbf{b}(P)$.

Suppose $a \in \mathbf{b}(S)$. Since $a \in \mathbf{b}(H)$ we have $a \notin P$. By a previous Theorem, $\mathbf{s}(b,c) \subset H$. Suppose $G \in \mathcal{F} \sim \{H\}$. Choose a point b in S. Then $\mathbf{s}(a,b) \subset S \subset G$. Since $b, c \in G$ and G is a halfspace $\mathbf{s}(b,c) \subset G$. Thus $\mathbf{s}(a,b) \subset G$ so $a \in \mathbf{b}(P)$.

Suppose $o \in \mathbf{b}(P)$. Choose a point d in p such that $\mathbf{s}(o,d) \subset P$. Choose a halfspace H in $\mathcal{F}$ such that $o \in \mathbf{b}(H)$; were such a choice impossible we would have $o \in P$. Let $S = \mathbf{s}(\mathcal{F}, H)$. I claim that $o \in S \cup \mathbf{b}(S)$. If this were not the case, there would be points a, b such that $\mathbf{b}(S) = \{a, b\}$ and $b \in \mathbf{s}(a, o)$. But, by the preceding Theorem, b would lie on $\mathbf{b}(G)$ for some $G \in \mathcal{F} \sim \{H\}$ and we would have $o \in G^{\mathbf{o}}$ which is incompatible with $\mathbf{s}(o,d) \subset P \subset G$.

Let

$$\mathcal{A} = \{\mathbf{s}(\mathcal{F}, H) : H \in \mathcal{F}\} \cup \{\cup\{\mathbf{b}(\mathbf{s}(\mathcal{F}, H)) : H \in \mathcal{E}\}\}.$$

We have shown that $F = \cup\mathcal{A}$. That $\mathcal{A}$ is disjointed follows directly from definitions and the fact that $\mathbf{b}(S) \cap S$ is empty whenever S is a a segment. □

Corollary. Suppose $\mathcal{E}_i,\ i = 1, 2$, are efficient families of halfspaces and $\cap\mathcal{E}_1 = \cap\mathcal{E}_2$. Then $\mathcal{E}_1 = \mathcal{E}_2$.
Proof. The families $\{\mathbf{s}(\mathcal{E}_i, H) : H \in \mathcal{E}_i\},\ i = 1, 2$, must be equal so the families $\{\mathbf{b}(H) : H \in \mathcal{E}_i\},\ i = 1, 2$, must be equal. It follows that $\mathcal{E}_1 = \mathcal{E}_2$. □
Corollary. Suppose F is a fundamental set. There is one and only one efficient family $\mathcal{E}$ of halfspaces such that $F = \cap\mathcal{E}$.

Definition. Suppose F is a fundamental set. Let $\mathcal{E}$ be that efficient family of halfspaces such that $F = \cap\mathcal{E}$. The members of $\mathcal{E}$ are called **bounding halfspaces for** F. The members of $\{\mathbf{s}(\mathcal{E}, H) : H \in \mathcal{E}\}$ are called **sides of** F. A point is a **vertex of** F if it is either an endpoint of a segment which is side of F or it is the origin of a ray which is a side of F. Note that

$$\{(S, H) : S \text{ is a side of } F,\ H \in \mathcal{E} \text{ and } S \subset \mathbf{b}(H)\}$$

is a univalent function which carries the set of sides of P onto the set of bounding halfspaces. Thus we may speak of the **side of** F **corresponding to a bounding halfspace** or the **bounding halfspace corresponding to a side**.

Theorem. Suppose F is a fundamental set, o is a vertex of F and

$$\mathcal{H} = \{H : H \text{ is a bounding halfspace for } F \text{ and } o \in \mathbf{b}(H)\}.$$

Then $\mathcal{H}$ has exactly two members. Furthermore, if S is the side of F corresponding to a member of $\mathcal{H}$ then either S is a segment and o is an endpoint of S or S is a ray and o the origin of S.
Proof. Were there three distinct bounding halfspaces H, I, J of F such that $\mathbf{b}(H) \cap \mathbf{b}(I) \cap \mathbf{b}(J) = \{o\}$ the family of bounding halfspaces of F, by virtue of a preceding Theorem, would not be efficient.

Let S be a side of F such that $o \in \mathbf{b}(S)$. Let H be that bounding halfspace for F such that $S \subset \mathbf{b}(H)$. By virtue of a preceding Theorem, there is bounding halfspace I of F such that $\mathbf{b}(H) \cap \mathbf{b}(I) = \{o\}$. Let T be the side of F contained in I. If $o \notin \mathbf{b}(T)$ then there are distinct points a, b such that $T = \mathbf{s}(a, b)$ and $a \in \mathbf{s}(o, b)$. Let J be the a bounding halfspace for F such that $a \in \mathbf{b}(J)$; such a bounding halfspace exists. Note that J cannot equal I or J. This implies that $S \cup T \subset J$ which is impossible since o and b are on opposite sides of $\mathbf{b}(J)$. □

Definition. Let F be a fundamental set.

Let o be a vertex of F. A side S of F is called **adjacent to** o if $o \in \mathbf{b}(S)$; otherwise it is called **opposite to** o. A vertex p of F not equal to o is called **adjacent to** o if $\mathbf{s}(o, p)$ is a side of F; otherwise it is called **opposite to** o.

The preceding Theorem says a vertex of a fundamental set has exactly two adjacent sides.

If o and p are opposite vertices of F the segment $\mathbf{s}(o, p)$ is called a **diagonal of** F.

Two distinct sides of F are **adjacent** if they have an endpoint in common; otherwise they are **opposite.**

Theorem. Suppose F is a fundamental set and a, b are distinct points in $\mathbf{b}(F)$. Then $\mathbf{s}(a, b) \subset F$ if and only if $\mathbf{s}(a, b)$ is not contained in any side of F.
Proof. Let $\mathcal{E}$ be the efficient family of halfspaces such that $F = \cap\mathcal{E}$

Suppose a and b are in different sides of F. Then there are distinct members H, I of $\mathcal{E}$ such that $a \in \mathbf{b}(H)$ and $b \in \mathbf{b}(I)$. Let G be the emptyset if $\mathcal{E} = \{H, I\}$ and let G equal $\cap\big(\mathcal{E} \sim \{H, I\}\big)$ otherwise. Then $a \in I \cap G$ and $b \in H \cap G$. It follows that $\mathbf{s}(a, b) \subset H \cap I \cap G = F$.

Suppose a is a vertex of F and b is in an opposite side of F. Then there are distinct members H, I, J of $\mathcal{E}$ such that $a \in \mathbf{b}(H) \cap \mathbf{b}(I)$ and $b \in \mathbf{b}(J)$. Let G be the emptyset if $\mathcal{E} = \{H, I, J\}$ and let G equal $\cap\big(\mathcal{E} \sim \{H, I, J\}\big)$ otherwise. Then $a \in J \cap G$ and $b \in H \cap I \cap G$. It follows that $\mathbf{s}(a, b) \subset H \cap I \cap J \cap G = F$.

Suppose a and b are opposite vertices of F. Then there are distinct members H, I, J, K of $\mathcal{E}$ such that $a \in \mathbf{b}(H) \cap \mathbf{b}(I)$ and $b \in \mathbf{b}(J)\mathbf{b}(K)$. Let G be the emptyset if $\mathcal{E} = \{H, I, J, K\}$ and let G equal $\cap\big(\mathcal{E} \sim \{H, I, J, K\}\big)$ otherwise. Then $a \in J \cap K \cap G$ and $b \in H \cap I \cap G$. It follows that $\mathbf{s}(a, b) \subset H \cap I \cap J \cap K \cap G = F$.
□

Definition. An **angle** is a fundamental set with one vertex and two sides By virtue of the preceding Theorem, these sides must be rays with origin equal to this vertex. By virtue of a preceding Theorem, if H and I are distinct halfspaces then $H \cap I$ is an angle if and only if $\mathbf{b}(H)$ meets $\mathbf{b}(I)$ in a point.

Theorem. Suppose R and S are distinct rays with origin o. Then there is one and only angle with sides R and S.
Proof. Choose a point a in R and a point b in S. Then $\mathbf{h}(\mathbf{l}(R), b) \cap \mathbf{h}(\mathbf{l}(S), a)$ is the desired angle. □

Definition. Suppose F is a fundamental set and v is a vertex of F. The **angle of F corresponding to the vertex** v is $H \cap I$ where H and I are the members of the efficient family of halfspaces whose intersection is F such that $\{v\} = \mathbf{b}(H) \cap \mathbf{b}(I)$.

Definition. A **polygon** is a fundamental set all of whose sides are segments. An **n-gon** is a polygon with n sides. A **triangle** is a 3-gon. A **quadrilateral** is a 4-gon. And so forth.

Definition. Suppose P is a an polygon and V is the set of its vertices. A **boundary path** of P is a function

$$v : \{0, 1, \ldots, l\} \to V$$

such that

(i) $l \geq 2$;

(ii) v_i and v_{i+1} are adjacent whenever $0 \leq i < l$;

(iii) $v_i \neq v_j$ if $0 \leq i < j < l$.

We say v **starts** at v_0 and **ends** at v_l.

Theorem. Suppose P is a n-gon, V is the set of vertices of P and a and b are are (not necessarily distinct) vertices of P. Then

(i) There are exactly two boundary paths

$$v : \{0, 1, \ldots, l\} \to V \quad \text{and} \quad w : \{0, 1, \ldots, m\} \to V$$

which start at a and end at b.

(ii) $l + m = n + 2$.

(iii) If $a = b$ then $l = m$ and $v_i = w_{n+1-i},\ i = 0, 1, \ldots, n+1$ and the ranges of v and w equal V.

(iv) If $a \neq b$ then $(\mathbf{rng}\, v \cap \mathbf{rng}\, w) \sim \{a, b\}$ is empty.

Proof. Simple combinatorics.

Theorem. The number of sides of a polygon equals the number of vertices.
Proof. Simple combinatorics. □

Theorem. A polygon has at least three sides.
Proof. Exercise for the reader. □

Theorem. Suppose a, b and c are distinct and $\{a, b, c\}$ is noncollinear. Let $H_a = \mathbf{h}(\mathbf{l}(b, c), a)$, $H_b = \mathbf{h}(\mathbf{l}(c, a), b)\}$ and $H_c = \mathbf{h}(\mathbf{l}(a, b), c)$.

Then $\{H_a, H_b, H_c\}$ is efficient and the intersection of these halfspaces is a triangle with vertices a,b and c with corresponding angles $H_b \cap H_c$, $H_a \cap H_c$, $H_a \cap H_b$.

Theorem. Suppose P is a polygon with at least four vertices; V is the set of vertices of P; and $v \in V$. Let u and w be the vertices of P such that $\mathbf{s}(u, v)$ and $\mathbf{s}(u, w)$ are the sides of P adjacent to v. let T be the triangle with vertices u, v and w; let $H = \mathbf{h}(\mathbf{l}(u, w), v)$; let $Q = P \cap H^{\mathbf{o}}$; let $\mathcal{E}$ be the efficient family of halfspaces whose intersection is P; and let

$$\mathcal{F} = (\mathcal{E} \sim \{\mathbf{h}(\mathbf{l}(u, v), w), \mathbf{h}(\mathbf{l}(u, v), w)\}) \cup \{H^{\mathbf{o}}\}.$$

Then $\{T, \mathbf{s}(u, w), Q\}$ is a partition of P; $\mathcal{F}$ is an efficient family of halfspaces; Q is a polygon with vertices $V \sim \{v\}$; and $Q = \cap\mathcal{F}$.
Remark This theorem can be used to prove other theorems about polygons by inducting on the number of vertices.
Proof. Exercise for the reader. □

Theorem. Two polygons with the same vertices are equal. In particular, there is one and only one triangle whose set of vertices is a given three point noncollinear set.
Proof. Use the previous Theorem. □

Theorem. Suppose P is a polygon and H is a halfspace. Then $P \subset H$ if and only if the set of vertices of P is a subset of $H \cup \mathbf{b}(H)$.
Proof. Suppose $P \subset H$, $a \in H^{\mathbf{o}}$ and a is a vertex of H. Let $b \in P$. We have shown earlier that $\mathbf{s}(a, b) \subset F$. But, as $b \in F \subset H$, $\mathbf{s}(a, b)$ would meet $\mathbf{b}(H)$.

Suppose the set of vertices of P is a subset of $H \cup \mathbf{b}(H)$. Let a, b, c be vertices of P such that a and b are the vertices adjacent to c. Let $I = \mathbf{h}(\mathbf{l}(a, c), b)$ and let $J = \mathbf{h}(\mathbf{l}(b, c), a)$. Then $F \subset I \cap J$, as J and H are members of the efficient family of halfspaces whose intersection is F. It is a simple matter which we leave to the reader to show that $I \cap J \subset H$. □

Theorem. Suppose P is a polygon, L is a line and $o \in L \cap P$. Then $L \cap P$ is a segment with endpoints in $\mathbf{b}(P)$.
Proof. Suppose no side of P meets L. Then there would be $H \in \mathbf{H}(L)$ such that the vertices of P are all in H. The previous Theorem would then imply that $P \subset H$ which is impossible because $o \in P \sim H$.

Let a, b be adjacent vertices of P such that $\mathbf{s}(a, b)$ meets L in a point c. Let v be that boundary path starting at a and ending at b whose range does not equal $\{a, b\}$ and which therefore has n members. Were it the case that $\mathbf{s}(v_i, v_{i+1}) \cap L$ were empty for each $i = 0, \ldots, n-1$ we would conclude by induction that $\mathbf{s}(v_0, v_i) \cap L$ is empty for each $i = 0, \ldots, n-1$ and this would imply that $\mathbf{s}(a, b) = \mathbf{s}(v_0, v_{n-1})$ did not meet L. So there is side $S = \mathbf{s}(d, e)$ of P not equal to $\mathbf{s}(a, b)$ which meets L in some point f. Since $o \in \mathbf{h}(\mathbf{l}(a, b), f) \cap \mathbf{h}(\mathbf{l}(d, e), c)$ we find that $o \in \mathbf{s}(c, f)$. □

Theorem. Suppose P is a polygon and $o \in P$. Then P is the disjoint union of $\{o\}$; the segments $\mathbf{s}(o, v)$ where v is a vertex of P; and the triangles with vertices o, a, b where $\mathbf{s}(a, b)$ is a side of P.
Proof. Use the previous Theorem and the the Crossbar Theorem. □

Theorem. Suppose P is a polygon and v is a boundary path whose range is the set of vertices of P. Suppose

$$0 \le i < j < k < l < n$$

where i, j, k, l are integers and $n \ge 4$ is the number of vertices of P. Then

$$\mathbf{s}(v_i, v_j) \cap \mathbf{s}(v_k, v_l) = \emptyset$$

and there is a point $o \in P$ such that

$$\mathbf{s}(v_i, v_k) \cap \mathbf{s}(v_j, v_l) = \{p\}.$$

Proof. Let $L = \mathbf{l}(v_k, v_l)$. Since none of the segments $\mathbf{s}(v_p, v_{p+1}),\ p = i, \ldots, j-1$ meet L we infer that v_i and v_j are on the same side of L and so $\mathbf{s}(v_i, v_j)$ does not meet L.

Let $M = \mathbf{l}(v_i, v_k)$. Arguing as we did in the preceding paragraph we find that the points $v_p,\ i < p < k$, are on the same side H of M and that the points $v_q,\ k < q < l$ or $l \le q < n$ or $0 < q < i$ are on the same side I of M. Were $H = I$ we would infer from a previous Theorem that $P \subset H$. This is impossible because $\mathbf{s}(v_i, v_k)$ meets P by a previous Theorem. Thus $I = H^{\mathbf{o}}$ and therefore $\mathbf{s}(v_j, v_l)$ meets M in a point p. In a previous Theorem we found that $\mathbf{s}(v_j, v_l) \subset P$. $\square$

More on angles.

Definition. Suppose H and I are halfspace that intersect in a point o. We set

$$\mathbf{a}(H, I) = H \cap I$$

and note that this set is an angle with vertex o and sides $\mathbf{b}(H) \cap I$ and $\mathbf{b}(I) \cap H$.

Suppose that if R and S are distinct rays with the same origin o. There is one and only one angle

$$\mathbf{a}(R, S)$$

whose sides are R and S. Its vertex is o.

If a, b, c are distinct points and $\{a, b, c\}$ is noncollinear we let

$$\mathbf{a}(a, b, c) = \mathbf{h}(\mathbf{l}(b, a), c) \cap \mathbf{h}(\mathbf{l}(b, c), a);$$

note that the sides of this angle are $\mathbf{r}(b, a)$ and $\mathbf{r}(b, c)$ and that its vertex is b.

Definition. Let A be an angle and let H and I be its bounding halfspaces. The angles $H^{\mathbf{o}} \cap I$ and $H \cap I^{\mathbf{o}}$ are **supplementary** to the angle $H \cap I$. The angle $H^{\mathbf{o}} \cap I^{\mathbf{o}}$ is **vertical** to the angle $H \cap I$. Note that these relations are symmetric because $J^{\mathbf{oo}} = J$ for any halfspace J.

Definition. Flags. A **flag** is an ordered pair (H, R) where H is a half space and R is a ray contained in the $\mathbf{b}(H)$.

Definition. Suppose (H, R) is a flag and o is the origin of R. We let

$$\mathbf{A}(H, R) = \{S \in \mathbf{R}(o) : S \subset H\}.$$

Theorem. Suppose o is a point; R is a ray with origin o; (H, R) is a flag; and

$$\prec$$

is the set of ordered pairs (S, T) such that $S, T \in \mathbf{A}(H, R)$ and

$$S \subset \mathbf{a}(R, T).$$

Then $\prec$ is a complete linear ordering of $\mathcal{S}$.

Proof. Suppose $S, T \in \mathcal{S}$ and $S \neq T$. Let I be the halfspace such that $S \subset \mathbf{b}(I)$ and $R \subset I$. Suppose $T \not\subset \mathbf{a}(R, S) = H \cap I$. Since $S \neq T$ and $T \subset H$ we have $T \subset I^{\mathbf{o}}$. Choose a point a in R and a point b in T. Since $a \in I$ and $b \in I^{\mathbf{o}}$ the segment $\mathbf{s}(a, b)$ must meet $\mathbf{b}(I)$ in a point c. Since $\mathbf{s}(a, b) \subset H$ we have $c \in S$. From the Crossbar Theorem we infer that $S = \mathbf{r}(o, c) \subset \mathbf{a}(R, T)$ so $S \prec T$. It follows that $\prec$ is trichotomous.

Suppose $S, T, U \in \mathcal{S}$, $S \prec T$ and $T \prec U$. Choose a point a in R and a point d in U. By the Crossbar Theorem,

$$\mathbf{a}(R, U) = \{\mathbf{r}(o, x) : x \in \mathbf{s}(a, d)\}.$$

Because $T \prec U$ there is a point $c \in T \cap \mathbf{s}(a, b)$. By the Crossbar Theorem,

$$\mathbf{a}(R, T) = \{\mathbf{r}(o, x) : x \in \mathbf{s}(a, c)\}.$$

Because $S \prec T$ there is a point $b \in S \cap \mathbf{s}(a, c)$ from which it follows that $S \subset \mathbf{a}(R, U)$. Thus $\prec$ is transitive.

Suppose $\mathcal{T}$ is a nonempty subset of $\mathcal{S}$ and u is a member of $\mathcal{S}$ which is an upper bound for $\mathcal{T}$. If $U \in \mathcal{T}$ then U is a least upper bound for $\mathcal{T}$ so let us suppose $U \notin \mathcal{T}$. Choose a point a in R and a point b in U. Let $<$ be the geometric linear ordering on $\mathbf{l}(a, b)$ such that $a < b$. By the Crossbar Theorem,

$$\mathbf{a}(R, U) = \{\mathbf{r}(o, x) : x \in \mathbf{s}(a, b)\}.$$

Let $\mathcal{U} = \{S \in \mathcal{S} : S < U\}$ and let f be such that $f : \{S \in \mathcal{S} : R < S < U\} \to \mathbf{s}(a, b)$ and $\mathbf{r}(o, f(S)) = T$ whenever $S \in \mathcal{U}$. The Crossbar Theorem implies that for any $S, T \in \mathcal{U}$ we have $f(S) < f(T)$ if and only if $S \prec T$. Let c be the least upper bound for the range of f and note that $c = b$ or $c \in \mathbf{s}(a, b)$. In case $c = b$ let $T = U$ and in case $c \in \mathbf{s}(a, b)$ we let $T \in \mathcal{U}$ be such that $f(T) = c$. T is a least upper bound for $\mathcal{U}$. $\square$

Definition. We call the ordering $\prec$ in the previous Theorem **the geometric linear ordering of $\mathbf{A}(H, R)$.**

Theorem. Suppose o is a point, $R, S, T, U \in \mathbf{R}(o)$ and $S, T \subset \mathbf{a}(R, U)$. Then

$$S \in \mathbf{a}(R, T) \;\Leftrightarrow\; T \in \mathbf{a}(S, U).$$

Proof. Straightforward exercise in the Crossbar Theorem. $\square$