Betweenness preserving permutations.

We let

\[ \tau \]

be the identity map of the set of points.

**Definition.** We say a permutation \( \tau \) of the set of points is betweenness preserving if

\[ \tau|_{s(a,b)} = s(\tau(a), \tau(b)) \]

whenever \( a \) and \( b \) are distinct points.

We let

\[ \mathcal{B} \]

be the set of betweenness preserving permutations of the set of points.

**Theorem.** \( \mathcal{B} \) is a subgroup of the group of permutations of the set of points.

**Proof.** Exercise. \( \square \)

**Remark.** We will use repeatedly the fact that if \( \tau \in \mathcal{B} \) and \( X \) is a set of points then

\[ \tau^{-1} [\tau[X]] = (\tau^{-1} \circ \tau)[X] = \iota[X] = X. \]

**Proposition.** Suppose \( A \) is a set of points and \( \tau \in \mathcal{B} \). Then

\[ \tau|_{\mathcal{b}(A)} = \mathcal{b}(\tau[A]). \]

**Proof.** Suppose \( y \in \tau|_{\mathcal{b}(A)} \). Then there is \( x \) in \( \mathcal{b}(A) \) such that \( y = \tau(x) \). Since \( x \in \mathcal{b}(A) \) there is \( a \) in \( A \) such that \( s(a, x) \subseteq A \). Let \( b = \tau(a) \) and note that \( b \in \tau[A] \). Since \( \tau \in \mathcal{B} \) we have \( \tau[s(a, x)] = s(\tau(a), \tau(x)) = s(b, y) \).

Were it the case that \( y \in \tau[A] \) we would have \( x = \tau^{-1}(y) \in \tau^{-1}[\tau[A]] = A \) which is impossible since \( x \in \mathcal{b}(A) \) implies \( x \not\in A \). Thus \( y \in \mathcal{b}(\tau[A]) \).

Suppose \( y \in \mathcal{b}(\tau[A]) \). Then \( y \notin \tau[A] \) and there is \( b \) in \( \tau[A] \) such that \( s(b, y) \subseteq \tau[A] \). Let \( x = \tau^{-1}(y) \) and let \( a = \tau^{-1}(b) \). Were it the case that \( x \in A \) we would have \( y = \tau(x) \in \tau[A] \) which is impossible. Moreover, \( a = \tau^{-1}(b) \in \tau^{-1}[\tau[A]] = A \). Since \( \tau^{-1} \in \mathcal{B} \) we have \( s(a, x) = s(\tau^{-1}(b), \tau^{-1}(y)) = \tau^{-1}[s(b, y)] \subseteq \tau^{-1}[\tau[A]] = A \). Thus \( x \in \mathcal{b}(A) \) so \( y = \tau(x) \in \mathcal{b}(\tau[A]) \). \( \square \)

**Theorem.** Suppose \( \tau \) is a betweenness preserving permutation of the set of points. The following statements hold:

(i) If \( a \) and \( b \) are distinct points then \( \tau|_{\mathcal{I}(a,b)} = \mathcal{I}(\tau(a), \tau(b)) \); in particular, if \( L \) is a line then \( \tau[L] \) is a line.

(ii) If \( o \) and \( a \) are distinct points then \( \tau|_{\mathcal{r}(o,a)} = \mathcal{r}(\tau(o), \tau(a)) \). In particular, if \( R \) is a ray then \( \tau[R] \) is a ray and \( \tau(\mathcal{o}(R)) = \mathcal{o}(\tau[R]) \).

(iii) If \( L \) is a line and \( a \) is a point not on \( L \). Then \( \tau(a) \) is not on \( \tau[L] \) and \( \tau|_{\mathcal{h}(L,a)} = \mathcal{h}(\tau[L], \tau(a)) \). In particular, if \( H \) is a half space then \( \tau[H] \) is a half space and \( \tau|_{\mathcal{b}(H)} = \mathcal{b}(\tau[H]) \).

(iv) If \( < \) is an geometric linear ordering of the line \( L \) then

\[ \{ (\tau(a), \tau(b)) : a, b \in L \text{ and } a < b \} \]

is a geometric ordering of the line \( \tau[L] \).

(v) If \( F \) is an fundamental set then \( \tau[F] \) is an fundamental set; if \( S \) is a side of \( F \) then \( \tau[S] \) is a side of \( \tau[F] \); if \( v \) is a vertex of \( F \) then \( \tau(v) \) is a vertex of \( \tau[F] \).
Proof. Exercise. □

Definition. Suppose $\tau \in B$, $L$ is a line and $\tau[L] = L$. As a consequence of (iv) in the preceding Theorem we see that either

$$\tau(a) < \tau(b) \iff a < b$$

whenever $<$ is a geometric linear ordering of $L$ and $a, b \in L$ in which case we say $\tau$ is order preserving on $L$ or

$$\tau(b) < \tau(a) \iff a < b$$

whenever $<$ is a geometric linear ordering of $L$ and $a, b \in L$ in which case we say $\tau$ is order reversing on $L$.

Here are three very important consequences of the Continuity Axiom.

Theorem. Suppose

(i) $\tau \in B$, $L$ is a line, $\tau[L] = L$ and $\tau$ is order preserving on $L$;

(ii) $\tau(a) \neq a$ for each $a \in L$; and

(iii) $o \in L$ and $<$ is the geometric linear ordering of $L$ such that $o < \tau(o)$.

Then for each $a \in L$ there is a unique integer $n$ such that

$$\tau^n(o) \leq a < \tau^{n+1}(o).$$

Moreover,

$$a < \tau(a) \quad \text{for} \quad a \in L.$$

Proof. Let $R = \{x \in L : o < x\}$. Let $A = \{\tau^n(o) : n \in \mathbb{N}\}$. Since $\tau$ preserves $<$ we find that $A \subset R$.

Suppose $a \in R$. By an earlier Theorem, $a$ is not an upper bound for $A$ so there is a least nonnegative integer $n$ such that $a < \tau^{n+1}(o)$. It follows that $\tau^n(o) \leq a$. Thus $\tau^n(o) \leq a < \tau^{n+1}(o)$. The uniqueness of $n$ follows directly from the fact that $\tau$ preserved $<$ as does the fact that $\tau^{n+1}(o) \leq \tau(a)$ so $a < \tau(a)$.

In case $a = o$ the assertion to be proved is trivial.

In case $a \in R^2$ we apply the results of the first paragraph with $\tau$ replaced by $\tau^{-1}$. □

Theorem. Suppose $\tau \in B$, $L$ is a line, $\tau[L] = L$ and $\tau$ is order reversing on $L$.

Then there is a point $o$ in $L$ such that $\tau(o) = o$.

Proof. This follows directly from earlier theory. □

Theorem. Suppose $(H, R)$ is a flag, $o$ is the origin of $R$, $\tau \in B$, $\tau(o) = o$,

$$R \subset \tau[H] \quad \text{and} \quad \tau[R] \subset H.$$

Then there is $S$ such that $S \in R(o)$, $S \subset a(R, \tau[R])$ and

$$\tau[S] = S.$$

Proof. Let $<$ be the geometric linear ordering on $A(H, R)$ and recall that, by a previous Theorem, $<$ is complete. We have $\tau^2[R] \subset \tau[R]$ so $\tau^2[H] \in A(H, R)$. Thus either $\tau^2[R] \subset a(\tau[R], R)$ or $R \subset a(\tau[R], \tau^2[R])$.

Suppose $(\tau^2[R] \subset a(\tau[R], R)$ holds. Let $T = \{S \in A(H, R) : S \prec R\}$. We have

$$S \in T \Rightarrow S \subset a(\tau[R], R) \Rightarrow \tau[S] \in a(\tau[R], \tau^2[R]) \Rightarrow \tau[S] \prec \tau^2[R] \Rightarrow \tau[S] < R \Rightarrow \tau[S] \in T.$$

Moreover, by an earlier Theorem,

$$S, T \in T \text{ and } S < T \Rightarrow S \subset a(\tau[R], T) \Rightarrow T \in a(R, S) \Rightarrow \tau[T] \subset a(\tau[R], \tau[S]) \Rightarrow \tau[T] < \tau[S].$$
Thus

\[ \{(S, \tau[S]) : S \in T \} \]

is order reversing function from \( T \) to itself. By earlier theory, there is a member \( S \) of \( T \) such that \( \tau[S] = S \).

Suppose \( R \subset a(\tau[R], \tau^2[R]) \) holds. Let \( U = \tau[R] \) and let \( I = \tau[H] \) and let \( \sigma = \tau^{-1} \). Then

\[ U \subset \sigma[I], \quad \sigma[U] \subset I, \quad \sigma^2[U] \subset a(\sigma[U], U) \]

so we may apply the result just obtained with \( \tau, R, H \) replaced by \( \sigma, U, I \) to secure \( S \in R(o) \) such that \( \sigma[S] = S \) and \( S \subset a(R, \tau[R]) \).

**Orientation.**

Let \( o \) be a point.

**Definition.** We let \( F(o) = \{(H, R) : (H, R) \text{ is a flag and } o = o(R)\} \).

**Definition.** Let

\[ K = \{(1,1), (1,-1), (-1,1), (-1,-1)\} \]

and let \( K^+ = \{(1,1), (-1,-1)\} \).

Note that

\[ (ac, bd) \in K \quad \text{whenever } (a, b), (c, d) \in K. \]

For each flag \( F = (H, R) \) and each \( (a, b) \in K \) we let

\[ F^{(a,b)} = \begin{cases} (H, R) & \text{if } (a, b) = (1,1); \\ (H, R^o) & \text{if } (a, b) = (1,-1); \\ (H^o, R) & \text{if } (a, b) = (-1,1); \\ (H^o, R^o) & \text{if } (a, b) = (-1,-1). \end{cases} \]

Note that

\[ F^{(a,b)}(c,d) = F^{(ac,bd)} \quad \text{whenever } (a, b), (c, d) \in K. \]

**Definition.** \( G = (I, S) \in F(o) \). We let

\[ p(G) = \{(H, R) \in F(o) : S \subset H \text{ and } R \subset I^o\}. \]

For each \( (a, b) \in K \) we let

\[ P^{(a,b)}(G) = \{ F \in F(o) : F^{(a,b)} \in p(G) \}. \]

**Proposition.** Suppose \( F, G \in F(o) \) and \( (a, b) \in K^+ \). Then

\[ F \in n(G) \iff F^{(a,b)} \in n(G^{(a,b)}). \]

**Proof.** The point here is that if \( S \in R(o) \) and \( H \) is a halfplane containing \( o \) then \( S \subset H \) if and only if \( S^o \subset H^o \).

**Proposition.** Suppose \( G \in F(o) \). Then

\[ \{ \{ G^{(a,b)} : (a, b) \in K \} \cup \{ p^{(a,b)} : (a, b) \in K \} \]
has eight members and is a partition of $F(o)$.

**Theorem.** Suppose $L_1, L_2, L_3$ are three distinct lines; $H_i \in H(L_i)$, $i = 1, 2, 3$; $L_1 \cap L_2 \cap L_3 \neq \emptyset$;

\[ L_1 \cap H_2 \cap H_3 \neq \emptyset \quad \text{and} \quad L_2 \cap H_2 \cap H_3^{\circ} \neq \emptyset. \]

Then \[ L_3 \cap H_1 \cap H_2 \neq \emptyset. \]

**Proof.** Let $a_1 \in L_1 \cap H_2 \cap H_3$ and let $a_2 \in L_2 \cap H_2 \cap H_3^{\circ}$. Then $s(a_1, a_2)$ is a subset of $H_1 \cap H_2$ and meets $L_3$ in a point $a_3$. Since $s(a_1, a_3) \subset s(a_1, a_2) \subset H_1$ we infer that $a_3 \in H_1$. Since $s(a_2, a_3) \subset s(a_1, a_2) \subset H_2$ we infer that $a_3 \in H_2$. \[ \square \]

**Definition.** For each $G \in F(o)$ we let

\[ o^+(G) = \{G(1,1)\} \cup \{G(-1,-1)\} \cup p^{(1,1)} \cup p^{(-1,-1)} \]

and we let

\[ o^{-}(G) = \{G(1,-1)\} \cup \{G(-1,1)\} \cup p^{(1,-1)} \cup p^{(-1,1)} \]

If $F, G \in F(o)$ we write

\[ F \sim_o G \]

and say $F$ has the **same orientation** as $G$ if $F \in o^+(G)$ and we write

\[ F \not\sim_o G \]

and say $F$ has the **opposite orientation** to $G$ if $F \in o^-(G)$.

**Theorem.** Suppose $F, G \in F(o)$. Then

\[ F \sim_o G \iff F^o \sim_o G^o. \]

**Proof.** Simple exercise for the reader. \[ \square \]

**Theorem.** Then $\sim_o$ is an equivalence relation on $F(o)$, corresponding to which there are exactly two equivalence classes.

Furthermore, if $G \in F(o)$ then $o^+(G)$ and $o^-(G)$ are the equivalence classes for $\sim_o$.

**Proof.** Suppose $(H_i, R_i)$, $i = 1, 2, 3$, are flags. We need to show that

1. \[(H_1, R_1) \sim_o (H_1, R_1); \]
2. \[(H_2, R_2) \sim_o (H_1, R_1) \Rightarrow (H_2, R_2) \sim_o (H_1, R_1); \]
3. \[(H_2, R_2) \sim_o (H_1, R_1) \text{ and } (H_3, R_3) \sim_o (H_2, R_2) \Rightarrow (H_3, R_3) \sim_o (H_1, R_1). \]

That (1) and (2) hold is obvious.

Let us prove (3). Suppose $(H_1, R_1) \in o^+( (H_2, R_2))$ and $(H_2, R_2) \in o^+( (H_3, R_3))$ but $(H_1, R_1) \in o^{-}( (H_3, R_3))$. Then \[ R_1 \subset H_2 \cap H_3; \quad R_2 \subset H_1^o \cap H_3; \quad R_3 \subset H_2^o \cap H_1. \]

Then \[ L_1 \cap H_3 \cap H_1 \neq \emptyset \quad \text{and} \quad L_3 \cap H_1 \cap H_2^o \neq \emptyset \] so the Lemma gives \[ L_2 \cap H_3 \cap H_1 \neq \emptyset \] which is incompatible with \[ R_2 \subset H_1^o \cap H_3. \]

**Definition.** Suppose $\tau \in B(o)$. We say a $\tau$ is **orientation preserving** if $(\tau[H], \tau[R]) \sim_o (H, R)$ for any $(H, R) \in F(o)$. We say a $\tau$ is **orientation reversing** if $(\tau[H], \tau[R]) \not\sim_o (H, R)$ for any $(H, R) \in F(o)$.

**Theorem.** Any member of $B(o)$ is either orientation preserving or orientation reversing. Furthermore, if $\sigma, \tau \in B(o)$ then $\tau \circ \sigma$ is orientation preserving if and only if $\sigma$ and $\tau$ are both orientation preserving or orientation reversing.

4