Angle sums and more.

Among other things, we will prove the following Theorems.

Theorem One. If the angle sum of some triangle is π then the angle sum of some right triangle is π.

Theorem Two. If the angle sum of some triangle is π then there is a rectangle.

Theorem Three. If there is a rectangle the the angle sum for every right triangle is π.

Theorem Four. If the angle sum for every right triangle is π then the angle sum for any triangle is π.

Theorem Five. If the angle sum for every right triangle is π then the parallel postulate holds.

Theorem Six. If the parallel postulate holds then the angle sum for every triangle is π.

The proofs follow. Along the way we will prove some Theorems which are interesting in their own right.

Definition. Suppose T is a triangle. We let

$$\delta(T)$$

be π minus the sum of the angles of T. Don't ever forget that, by the Saccheri Legendre Theorem, $\delta(T) \geq 0$. For some reason that escapes me the book calls $\delta(T)$ the **defect of** T. Does that mean there might be something wrong with T if its "defect" $\delta(T)$ is positive?

Theorem. The "additivity of the defect" in the book. Suppose T is a triangle with vertices a, b, c and $d \in \mathbf{s}(b, c)$. Let T' and T'' be the triangles with vertices a, b, d and a, c, d, respectively. Then

$$\delta(T) = \delta(T') + \delta(T'').$$

Proof. Let A' and A'' be the angles of T' and T'', respectively, corresponding to the vertex a. Let D' and D'' be the angles of T' and T'', respectively, corresponding to the vertex d. By the Crossbar Theorem, $|A| = |A'| + |A''|$. Thus

$$\begin{aligned}
\delta(T) &= \pi - (|A| + |B| + |C|) \\
&= \pi - (|A'| + |A''| + |B| + |C|) \\
&= \pi - (|A'| + |A''| + |B| + |C|) + (\pi - (|D'| + |D''|) \\
&= (\pi - (|A'| + |B| + |D'|)) + (\pi - (|A'| + |C| + |D''|)) \\
&= \delta(T') + \delta(T'').
\end{aligned}$$

□

Theorem. Suppose T is a triangle, a is a vertex of T, A is the angle corresponding to a and d is the perpendicular dropped from a to the line containing the opposite side.

If A is not acute then d is in the opposite side.

If d is not in the opposite side and d is not a vertex of T then A is acute and there is exactly one angle B of T such that B is obtuse.

Proof. If d is a vertex of T then A is acute because the angle of T corresponding to d is a right angle and the angle sum of T does not exceed π.

Suppose d is not in the side of T opposite a and d is not a vertex of T. Then there are b, c such that a, b, c are the vertices of T and $b \in \mathbf{s}(c, d)$. Let B be the angle corresponding to b. Since the triangle with vertices a, d, b has a right angle at d and since its angle sum does not exceed π we infer that $\mathbf{a}(d, b, a)$ is acute. Thus B is obtuse. Finally, by the exterior angle theorem, A is acute. □

Remark. We needed this for the Pythagorean Theorem. See how easy it is for a proof in this subject to have a gap?

Proof of Theorem One. Let T be a triangle with angle sum π. Then, by the preceding Theorem, there is a vertex a of T such that the perpendicular dropped from a to the line containing the opposite side is in the opposite side. By the Theorem on the "additivity of the defect" we obtain two right triangles with defect zero. $\square$

Proof of Theorem Two. Let T be a right triangle with zero defect. Let R be the union of T with one of its sides and the reflection across the line containing that side. Make sure you check that you get a quadrilateral! $\square$

Theorem. Suppose Q is a quadrilateral; a, b, c, d are the distinct vertices of Q;

$$\mathbf{s}(a,b), \quad \mathbf{s}(c,d), \quad \mathbf{s}(a,c), \quad \mathbf{s}(b,c)$$

are the sides of Q; A, B, C, D are the angles of Q corresponding to a, b, c, d; and

$$|A| = |B|.$$

Then

$$|C| = |D| \;\Leftrightarrow\; |\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|;$$

$$|C| < |D| \;\Leftrightarrow\; |\mathbf{s}(a,c)| > |\mathbf{s}(b,d)|;$$

$$|D| < |C| \;\Leftrightarrow\; |\mathbf{s}(a,c)| < |\mathbf{s}(b,d)|.$$

Proof. Let ρ be reflection across the perpendicular bisector of $\mathbf{s}(a,b)$. Note that ρ interchanges $R = \mathbf{r}(a,c)$ and $S = \mathbf{r}(b,d)$. (Why?)

If $|\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|$ then $\rho(c) = d$ and $\rho(d) = c$. (Why?) Thus ρ interchanges C and D so $|C| = |D|$. Thus

(1) $$|\mathbf{s}(a,c)| = |\mathbf{s}(b,d)| \;\Rightarrow\; |C| = |D|.$$

If $|\mathbf{s}(a,c)| < |\mathbf{s}(b,d)|$ then $|\mathbf{s}(b,\rho(c))| = |\rho[\mathbf{s}(a,c)]| < |\mathbf{s}(b,d)|$ so, as $\rho(c) \in S$, $d \in \mathbf{s}(b,\rho(c))$. From our earlier theory on polygons we know that $\mathbf{s}(c,\rho(c)) \subset Q$. From the exterior angle theorem we infer that

$$|C| > |\mathbf{a}(a,c,\rho(c))| = |\rho[\mathbf{a}(a,c,\rho(c))]| = |\mathbf{a}(b,\rho(c),c)]| > |\mathbf{a}(c,d,\rho(c))| = |D|.$$

Thus

(2) $$|\mathbf{s}(a,c)| < |\mathbf{s}(b,d)| \;\Rightarrow\; |D| < |C|.$$

Permuting a, b, c, d appropriately we infer from (2) thatthat

(3) $$|\mathbf{s}(b,d)| < |\mathbf{s}(a,c)| \;\Rightarrow\; |C| < |D|.$$

Suppose $|C| = |D|$. By the results of preceding two paragraphs we infer that neither $|\mathbf{s}(a,c)| < |\mathbf{s}(c,d)|$ nor $|\mathbf{s}(c,d)| < |\mathbf{s}(a,c)|$ holds. By trichotomy of segment ordering we conclude that $|\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|$. Thus

(4) $$|C| = |D| \;\Rightarrow\; |\mathbf{s}(a,c)| = |\mathbf{s}(b,d)|.$$

Suppose $|C| < |D|$. By (1) $\mathbf{s}(a,b)$ is not congruent to $\mathbf{s}(c,d)$. By trichotomy of segment ordering either $|\mathbf{s}(a,c)| < |\mathbf{s}(b,d)|$ or $|\mathbf{s}(b,d)| < |\mathbf{s}(a,c)|$. The first of these is excluded by (2). Thus

(5) $$|C| < |D| \;\Rightarrow\; |\mathbf{s}(b,c)| < |\mathbf{s}(a,c)|$$

Permuting a, b, c, d appropriately we infer from (5) thatthat

(5) $$|D| < |C| \;\Rightarrow\; \mathbf{s}(a,d) < \mathbf{s}(b,c).$$

□

Corollary. Opposite sides of a rectange are congruent.
Proof. One can take all of A, B, C, D to be right angles above. □

Proof of Theorem Three. Let R be a rectangle with vertices a, b, c, d such that $\mathbf{s}(a, b) \simeq \mathbf{s}(c, d)$ and $\mathbf{s}(a, c) \simeq \mathbf{s}(b, d)$.

Let T be a triangle with vertices e, f, g and with the angle corresponding to e being a right angle. Choose positive integers m and n such that $m|\mathbf{s}(a, b)| \geq < \mathbf{s}(e, f) >$ and $n|\mathbf{s}(a, c)| \geq |\mathbf{s}(e, d)|$.

Let $\tau_{a,b}$ be translation from a to b and let $\tau_{a,c}$ be translation from a to c.

$$a' = a. \quad b' = \tau_{a,b}^m(a); \quad c' = \tau_{a,c}^n(c); \quad d' = \tau_{a,c}^n(\tau_{a,b}^m(a)).$$

We leave it to the reader to use the above Corollary and our theory developed to this point to show that a', b', c', d' are the vertices of a rectangle R'.

Let $e' = a'$; let $d' \in \mathbf{s}(a', b')$ be such that $\mathbf{s}(e', d') \simeq \mathbf{s}(e, d)$; and let $f' \in \mathbf{s}(a', c')$ be such that $\mathbf{s}(e', f') \simeq \mathbf{s}(e, f)$. Let T' be the triangle with vertices d', e', f'. Evidently, $T \simeq T'$.

Since $\mathbf{s}(c', b')$ is a diagonal of R', the triangle with vertices e', b', c' has zero defect. By the additivity of the defect so does the triangle with vertices e', c', c'. By the additivity of the defect so does T'. □

Proof of Theorem Four. Just drop an appropriate perpendicular and use the additivity of the defect.

Aristotle's Axiom. Suppose R, S are the sides of an acute angle and l is a positive real number. There is a point p on R such that if q is the foot of the perpendicular dropped from p to the line containing S then $|\mathbf{s}(p, q)| > l$.
Proof. Let o be the vertex of $\mathbf{a}(R, S)$. Let $e \in R$, let b be the foot of the perpendicular dropped from e to the line containing S and note that $b \in S$. (Why?) Let α_a be the half turn about e. Let $c = \alpha_a(o)$, let d be the foot of the perpendicular dropped from c to the line containing S and let $a = \alpha_e(b)$. Note that a, b, c, d are the vertices of a quadrilateral. Let A, B, C, D be the corresponding angles. Note that A, B, D are right angles. Thus C has measure at most $\pi/2$. By the previous Theorem, $2|\mathbf{s}(e, b)| = |\mathbf{s}(a, b)| \leq |\mathbf{s}(c, d)|$.

By iterating this construction and appealing the Archimedean Property of segment ordering we obtain the desired point p on R. □
Remark. Note that, again by the previous Theorem, $|\mathbf{s}(b, d)| \leq |\mathbf{s}(a, c)| = |\mathbf{s}(o, b)|$.

We're now ready to prove Theorem Five.
Proof of Theorem Five. Suppose L is a line and $p \notin L$. Let $o \in L$ be the foot of the perpendicular dropped from p to L. Let M be the line through p making a right angle with $\mathbf{l}(o, p)$. Note that M is parallel to L by the alternating interior angle theorem. Suppose O were another parallel to L through p. Then $O \subset H = \mathbf{h}(L, p)$. Let $I = \mathbf{h}(M, o)$. Let $R = I \cap O$ and note that R is a ray with origin p and $R \subset H \cap I$. Use Aristotle's Axiom to obtain points $r \in O$ and $s \in M$ such that s is the foot of the perpendicular dropped from r to s and $|\mathbf{s}(r, s)| > |\mathbf{s}(o, p)|$. Let q be the foot of the perpendicular dropped from r to $\mathbf{l}(o, p)$ and note that $r \in \mathbf{s}(o, p)$. Note that p, q, r, s are the vertices of a quadrilateral Q. Since r, s, p and r, q, p are the vertices of right triangles we find that Q is a rectangle. Since $\mathbf{s}(p, q)$ and $\mathbf{s}(r, s)$ are opposite sides of this rectangle we infer from a Theorem above that $|\mathbf{s}(p, q)| = |\mathbf{s}(r, s)|$ which is incompatible with $|\mathbf{s}(p, q)| < |\mathbf{s}(o, p)| < |\{Sr, s|$.
□

We have already proved Theorem Six.