The average was 45.8. The standard deviation was 16.71.

1. **5 points.** Let \( f(z) = \sin \frac{z}{z} \) for \( z \in \mathbb{C} \sim \{0\} \). Is 0 a removable singularity?

   **Solution.**

   \[
   \sin \frac{z}{z} = \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) = 1 - \frac{z^2}{3!} + \frac{z^2}{5!} - \cdots
   \]

   so 0 is a removable singularity.

2. Let \( f(z) = \frac{z}{(z-1)(z-2)^2}, \ z \in \mathbb{C} \sim \{0\} \).

   (a) **(15 points.)** Identify the poles of \( f \) and their orders and calculate the residues.

   (b) **(5 points.)** Calculate \( \int_C f(z) \, dz \) where \( C \) is the circle of radius 3 centered at the origin which winds clockwise about the origin.

   We have

   \[
   f(z) = \frac{1}{z-1} f_1(z) \quad \text{where} \quad f_1(z) = \frac{z}{(z-2)^2}
   \]

   and \( f_1(1) = 1 \neq 0 \) so 1 is a pole of order 1 and the residue is \( f_1(1) = 1 \). We have

   \[
   f(z) = \frac{1}{(z-2)^2} f_2(z) \quad \text{where} \quad f_2(z) = \frac{z}{z-1}
   \]

   and \( f_2(2) = 2 \neq 0 \) so 2 is a pole of order 2 and the residue is \( f_2'(2) = -1 \).

   The integral is

   \[-2\pi i (\text{Res}(f, 1) + \text{Res}(f, 2)) = -2\pi i (1 - 1) = 0;\]

   the leading minus is because the curve is going clockwise.

3. **10 pts.** Let \( f(z) = e^{1/z} \) for \( z \in \mathbb{C} \sim \{0\} \). There exist a coefficient sequence \( c_0, c_1, c_2, \ldots \) and positive radii \( R \) such that

   \[
   f(z) = \sum_{n=0}^{\infty} c_n (z-1)^n, \quad |z-1| < R.
   \]

   Identify all such \( R \) and calculate \( c_0, c_1, c_2 \).

   **Solution.** \( f \) is analytic in a disc of radius 1 centered at 1 and in no larger disc so \( R = 1 \). Moreover,

   \[
   c_0 = f(1) = e; \quad c_1 = f'(1) = -e; \quad c_2 = \frac{f''(1)}{2!} = \frac{3e}{2}.
   \]

4. **10 points.** Suppose \( a, b, c \) are real numbers such that \( a \neq 0 \) and \( b^2 - 4ac < 0 \). Calculate

   \[
   \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}.
   \]
Solution. Let $f(z) = \frac{1}{az^2 + bz + c}$. Let $D = 4ac - b^2 > 0$, let

$$r_+ = \frac{-b + i\sqrt{D}}{2a} \quad \text{and let} \quad r_- = \frac{-b - i\sqrt{D}}{2a}.$$

Then

$$f(z) = \frac{1}{a(z - r_+)(z - r_-)}$$

so $f$ has simple poles at $r_+$ and $r_-$ with respective residues

$$\frac{1}{a(r_+ - r_-)} = \frac{1}{i\sqrt{D}} \quad \text{and} \quad \frac{1}{a(r_- - r_+)} = \frac{1}{i\sqrt{D}}.$$

Suppose $R > |r_+|$. Let $I_R$ be the segment from $(-R, 0)$ to $(R, 0)$ and let $S_R$ be the semicircle of radius $R$ centered at the origin which goes from $(R, 0)$ to $(-R, 0)$ in the upper half plane. By the residue theorem,

$$\int_{I_R+S_R} f(z) \, dz = 2\pi i \frac{1}{i\sqrt{D}} = \frac{2\pi}{\sqrt{D}}.$$

Since $\lim_{|z| \to \infty} |z||f(z)| = 0$ we find that

$$\lim_{R \to \infty} \int_{S_R} f(z) \, dz = 0.$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \lim_{R \to \infty} \int_{I_R} f(z) \, dz = \lim_{R \to \infty} \int_{I_R+S_R} f(z) \, dz = \frac{2\pi}{\sqrt{D}}.$$

5. Let $f$ be the $2\pi$ periodic function such that

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0, \\ x & \text{if } 0 \leq x < \pi/2, \\ 0 & \text{if } \pi/2 \leq x < \pi. \end{cases}$$

(a) (10 points) Compute $(f, E_n)$, $n$ any integer. (I suggest you use the jump formula.)

(b) (10 points) Evaluate

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (f, E_n)i^n.$$

Solution.

$$(f, E_0) = \int_{-\pi}^{\pi} f(x) \, dx = \frac{x^2}{2} \bigg|_{x=\pi/2}^{x=\pi} = \frac{\pi^2}{8}.$$ 

For $n \neq 0$ we have, as $f$ has a jump of $-\pi/2$ at $x = \pi/2$

$$(f, E_n) = \frac{1}{in} \left( (f', E_n) + (\pi/2)e^{-in\pi/2} \right) = \frac{1}{in} \left( (f', E_n) - \left(\frac{\pi}{2}\right)(-i)^n \right).$$

As $f'$ had a jump of 1 at $x = 0$ and $-1$ at $x = \pi/2$ we have

$$(f', E_n) = \frac{1}{i\pi n} \left( (f'', E_n) + e^{-in0} - e^{-in\pi/2} \right) = \frac{1}{i\pi n} \left( 1 - (-i)^n \right).$$
Thus if \( n \neq 0 \) we have
\[
(f, E_n) = \frac{1}{in} \left( \frac{1}{in} (1 - (-i)^n) - \left( \frac{\pi}{2} \right)(-i)^n \right).
\]

By the Fourier inversion formula the answer to (b) is the average value of \( f \) at \( \pi/2 \) since \( i^n = E_n(\pi/2) \). This average value is \( \pi/4 \).

6. 10 points. Let \( f(x) = 1 - x \) for \( 0 < x < 1 \) and let
\[
a_n = 2 \int_0^1 f(x) \cos n\pi x \, dx, \quad n = 1, 2, 3, \ldots
\]
Determine
\[
A(x) = \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{and} \quad B(x) = \sum_{n=1}^{\infty} n\pi a_n \sin n\pi x
\]
for \( x \in (0, 1) \).

**Solution.** Suppose \( 0 < x < 1 \). Then \( f(x) = \frac{a_0}{2} + A(x) \). Since \( a_0 = 2 \int_0^1 f(x) \, dx = 1 \) we find that \( A(x) = \frac{1}{2} - x \). Note that \( B(x) \) is minus term-by-term differentiation \( A(x) \); thus \( B(x) = 1 \).

7. Suppose \( a > 0 \) and \( b \in \mathbb{R} \). Let \( f(t) = e^{-a|t|} \sin bt \) for \( t \in \mathbb{R} \).

(a) (10 points.) Calculate \( \hat{f} \). (I suggest you begin by expressing \( f \) in terms of complex exponentials.)

(b) (10 points.) Evaluate
\[
\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega} \, d\omega.
\]

**Solution.** We have
\[
2i\hat{f}(\omega) = 2i \int_{-\infty}^{\infty} e^{-a|t|} \sin bt \, e^{-i\omega t} \, dt
\]
\[
= \int_{-\infty}^{\infty} e^{-a|t|} (e^{ibt} - e^{-ibt}) e^{-i\omega t} \, dt
\]
\[
= \int_{-\infty}^{0} e^{at+i(b-\omega)t} \, dt + \int_{0}^{\infty} e^{-at+i(b-\omega)t} \, dt
\]
\[
= \frac{e^{at+i(b-\omega)t}}{a+i(b-\omega)} \bigg|_{-\infty}^{0} + \frac{e^{-at+i(b-\omega)t}}{a+i(b-\omega)} \bigg|_{0}^{\infty} + \frac{e^{-at+i(-b-\omega)t}}{a+i(-b-\omega)} \bigg|_{-\infty}^{0} + \frac{e^{at+i(-b-\omega)t}}{a+i(-b-\omega)} \bigg|_{0}^{\infty}
\]
\[
= \frac{1}{a+i(b-\omega)} - \frac{1}{a+i(b-\omega)} + \frac{1}{a+i(-b-\omega)} - \frac{1}{a+i(-b-\omega)}
\]
\[
= \frac{2a}{a^2 + (b-\omega)^2} + \frac{2a}{a^2 + (b+\omega)^2}.
\]

By the Fourier inversion formula,
\[
\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega} \, d\omega = 2\pi f(1) = 2\pi e^{-a} \sin b.
\]