Theorem. Suppose f is a $2L$-periodic function, n is a nonzero integer, M is a positive integer,

$$-L \le a_1 < a_2 < \cdot < a_M < L,$$

$$a_0 = a_M - 2L,$$

f is continuously differentiable on the intervals

$$(a_0, a_1), (a_1, a_2), \ldots, (a_{M-1}, a_M)$$

and the limits

$$\lim_{x \uparrow a_j} f'(x), \quad \lim_{x \downarrow a_j} f'(x)$$

exist for $j = 1, \ldots, M$.

Then the limits

$$\lim_{x \uparrow a_j} f(x), \quad \lim_{x \downarrow a_j} f(x)$$

exist for $j = 1, \ldots, M$ and

$$(f, E_n) = \frac{L}{i\pi n}\left((f', E_n) + \sum_{j=1}^{M} J_m \overline{E_n(a_m)}\right)$$

where

$$J_j = \lim_{x \downarrow a_j} f(x) - \lim_{x \uparrow a_j} f(x), \quad j = 1, \ldots, M.$$

Proof. For any $j = 1, \ldots, M$ we have

$$\begin{aligned}
&\int_{a_{j-1}}^{a_j} f(x) e^{-in\pi x/L}\, dx \\
&\quad = \int_{a_{j-1}}^{a_j} f(x) d\left(\frac{e^{-in\pi x/L}}{-in\pi/L}\right) dx \\
&\quad = f(x)\left(\frac{e^{-in\pi x/L}}{-in\pi/L}\right)\Bigg|_{a_{j-1}}^{a_j} - \int_{a_{j-1}}^{a_j} f'(x)\left(\frac{e^{-in\pi x/L}}{-in\pi/L}\right) dx \\
&\quad = \frac{L}{in\pi}\left(\lim_{x \downarrow a_{j-1}} f(x)\overline{E_n(a_{j-1})} - \lim_{x \uparrow a_j} f(x)\overline{E_n(a_j)} + \int_{a_{j-1}}^{a_j} f'(x) e^{-in\pi x/L}\, dx\right).
\end{aligned}$$

Now sum over $j = 1, \ldots, M$. □