Matrices, vectors and covectors.

Let $F$ be a field (e.g. the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$). In linear algebra the members of $F$ are called scalars.

**Definition.** Let $m, n$ be positive integers and We let $F_{m \times n}$ be the set of rectangular arrays

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix},$$

where $a_{ij}$, which is found in the $i$-th row and $j$-th column, is in the field $F$. We call such an array an $m$ by $n$ matrix (with entries in $F$); the ordered pair $(i, j)$ is the shape of the matrix.

More precisely, an $m$ by $n$ matrix is a function whose domain is the Cartesian product $\{1, \ldots, m\} \times \{1, \ldots, n\}$ and whose range is a subset of $F$.

Using $a_{ij}$ to denote the entry of $A$ in the $i$-th row and $j$-th column, while quite common, is highly ambiguous; for example, if $m = n = 234$ then $a_{1234}$ has three different interpretations; it could be the entry in the first row and 234-th column; or the entry in the 12-th row and 34-th column; or the entry in the 123-rd row and 4-th column. In addition there is no logical relationship between $A'$ and $a'$. A technically better notation, which we will use when appropriate, is to let $A_{ij}$ be the element in the $i$-th row and $j$-th column whenever $A \in F_{m \times n}$.

Oftentimes we will not distinguish between a scalar $c \in F$ and the member $[c]$ of $F_1$.

**The zero matrix.** Whenever $m$ and $n$ are positive integers we let

$$O$$

be the $m$ by $n$ matrix such that

$$O_{ij} = 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

We call $O$ the zero matrix.

**Scalar multiplication.** Whenever $m$ and $n$ are positive integers, $c \in F$ and $A \in F_{m \times n}$ we let

$$cA \in F_{m \times n}$$

be such that

$$(cA)_{ij} = cA_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$  

We call $cA$ scalar multiplication (of the matrix $A$ by the scalar $c$).

**Matrix addition.** Whenever $m$ and $n$ are positive integers and $A, B \in F_{m \times n}$ we let

$$A + B \in F_{m \times n}$$
be such that
\[(A + B)_{ij} = A_{ij} + B_{ij}\] whenever \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\).

We call \(A + B\) the \textbf{(matrix) sum of} \(A\) and \(B\).

**The negative of a matrix.** Whenever \(m\) and \(n\) are positive integers and \(A, B \in \mathbb{F}^{m \times n}\) we let

\[-A \in \mathbb{F}^{m \times n}\]

be such that
\[(-A)_{ij} = -(A_{ij}), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.\]

We let
\[A - B = A + (-B).\]

**Some properties of scalar multiplication and matrix addition.** We leave it to the reader to provide all the hypotheses. We have

(i) \((A + B) + C = A + (B + C)\);
(ii) \(A + O = A = O + A\);
(iii) \(A + (-A) = O = (-A) + A\);
(iv) \(A + B = B + A\);
(v) \(c(A + B) = cA + cB\);
(vi) \((c + d)A = cA + dA\);
(vii) \((cd)A = c(dA)\);
(viii) \(1A = A\);

**Diagonal matrices.** Let \(n\) be a positive integer. Whenever \(c_1, c_2, \ldots, c_n\)
is an array of \(n\) scalars we let

\[
\text{diag}(c_1, c_2, \ldots, c_n) = \begin{bmatrix}
    c_1 & 0 & \cdots & 0 \\
    0 & c_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & c_n
\end{bmatrix} \in \mathbb{F}^{n \times n}.
\]

We note that
\[O = \text{diag}(0, 0, \ldots, 0).\]

**Identity matrices.** We let
\[I = \text{diag}(1, 1, \ldots, 1)\]
and call \(I\) the \textbf{n by n identity matrix}. We will follow the custom of letting
\[
\delta^i_j = I^i_j = \begin{cases}
1 & \text{if } i = j, \\
0 & \text{else}
\end{cases}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.
\]
Matrix multiplication. Whenever $l, m$ and $n$ are positive integers, $A \in \mathbb{F}^l_m$ and $B \in \mathbb{F}^m_n$ we let

$$AB \in \mathbb{F}^l_n$$

be such that

$$(AB)_j^i = \sum_{k=1}^{m} A^i_k B^k_j$$

whenever $i = 1, \ldots, l$, $j = 1, \ldots, n$.

We call $AB$ the (matrix) product of $A$ and $B$.

Example. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and let} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. $$

So $A, B \in \mathbb{Q}_2^2$. We have

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$AB \neq BA.$$

In fact, in a sense that can be made precise, $AB \neq BA$ for almost all $A, B$.

Properties of matrix multiplication. We leave it to the reader to supply all the hypotheses. We have

(i) $A(B + C) = AB + AC$;

(ii) $(A + B)C = AC + BC$;

(iii) $AI = A = IA$;

(iv) $c(AB) = (cA)B = A(cB)$;

(v) $A(BC) = (AB)C$.

Proof. These are all rather obvious except, perhaps, (v). To prove it we suppose $l, m, n, p$ are positive integers, $A \in \mathbb{F}^l_m$, $B \in \mathbb{F}^m_n$, $C \in \mathbb{F}^n_o$ and, with $1 \leq i \leq l$ and $1 \leq j \leq o$, we calculate

$$(A(BC))_j^i = \sum_{k=1}^{m} A^i_k (BC)_j^k = \sum_{k=1}^{m} A^i_k \left( \sum_{l=1}^{n} B^k_l C^l_j \right)$$

as well as

$$((AB)C)_j^i = \sum_{l=1}^{n} (AB)_j^l C^l_j = \sum_{l=1}^{n} \left( \sum_{k=1}^{m} A^l_k B^k_l \right) C^l_j;$$

and note that, by laws of arithmetic (which ones?) the rightmost terms are equal. \( \square \)

The transpose of a matrix. Whenever $m$ and $n$ are positive integers and $A \in \mathbb{F}^m_n$ we let

$$A^t \in \mathbb{F}^n_m$$

be such that

$$(A^t)_j^i = A^i_j$$

whenever $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

We call $A^t$ the (matrix) transpose of $A$.

Properties of the transpose. We leave it to the reader to supply all the hypotheses. We have
(i) \((A + B)^t = A^t + B^t\);
(ii) \((cA)^t = cA^t\);
(iii) \((A^t)^t = A\);
(iv) \(O^t = O\);
(v) \(I^t = I\);
(vi) \((AB)^t = B^tA^t\).

These are all obvious except perhaps for (vi). Suppose \(l, m, n\) are positive integers, \(A \in \mathbf{F}^l_m\) and \(B \in \mathbf{F}^m_n\). Then if \(1 \leq i \leq l\) and \(1 \leq j \leq n\)
\[ ((AB)^t)_i^j = (AB)_j^i = \sum_{k=1}^{m} A_k^i B_j^k \]
and
\[ (B^tA^t)_i^j = \sum_{k=1}^{m} (B^t)_k^i (A^t)_i^k = \sum_{k=1}^{m} B_j^k A_i^k. \]

**Vectors.** Suppose \(n\) is a positive integer. We let
\[ \mathbf{F}^n = \mathbf{F}^n_1. \]
The members of \(\mathbf{F}^n\) are called \(n\)-vectors or \((n-)\)column vectors. Whenever \(X \in \mathbf{F}^n\) and \(j = 1, \ldots, n\) we let
\[ X^j = X^j_1 \in \mathbf{F}. \]
For each \(j = 1, \ldots, n\) we let
\[ E_j \]
be that member of \(\mathbf{F}^n\) such that
\[ (E_j)_1^i = \delta_j^i, \quad i = 1, \ldots, n; \]
it is called the \(j\)-th standard basis vector in \(\mathbf{F}^n\).

**Covectors.** We let
\[ \mathbf{F}_n = \mathbf{F}^1_n. \]
The members of \(\mathbf{F}_n\) are called \(n\)-covectors or \((n-)\)row vectors. Whenever \(Y \in \mathbf{F}_n\) and \(i = 1, \ldots, n\) we let
\[ Y_i = Y_i^1 \in \mathbf{F}. \]
For each \(i = 1, \ldots, n\) we let
\[ E^i \]
be that member of \(\mathbf{F}_n\) such that
\[ (E^i)_j^1 = \delta_j^i, \quad j = 1, \ldots, n; \]
it is called the \(j\)-standard basis covector in \(\mathbf{F}_n\).

**Definition.** Suppose \(m\) and \(n\) are positive integers. Whenever \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\) we let
\[ E_j^i = E_j E^i \]
where \(E_j\) is the \(j\)-th standard basis vector in \(\mathbf{F}^m\) and \(E_i\) the \(i\)-th standard basis covector in \(\mathbf{F}_n\); thus
\[ (E_j^i)_l^k = \delta_l^j \delta_j^k, \quad k = 1, \ldots, m, \quad l = 1, \ldots, n. \]
Thus $E^i_j$ has 1 in its $j$-th row and $i$-th column which might not be what you expected; get used to this.

Note that $E^i_j E_j = [\delta^i_j]$.

**Proposition.** Suppose $m$ and $n$ are positive integers and $A \in \mathbb{F}_n^m$. Then

$$A = \sum_{i=1}^m \sum_{j=1}^n A^i_j E^j_i.$$ 

**Proof.** Whenever $k = 1, \ldots, m$ and $l = 1, \ldots, n$ we have

$$\left( \sum_{i=1}^m \sum_{j=1}^n A^i_j E^j_i \right)_k^l = \sum_{i=1}^m \sum_{j=1}^n A^i_j (E^j_i)_k^l = \sum_{i=1}^m \sum_{j=1}^n A^i_j \delta^i_j \delta^l_k = \delta^l_k.$$ 

Make sure you understand each step! \(\Box\)

**Definition.** Suppose $m$ and $n$ are integers and $A \in \mathbb{F}_n^m$. For each $i = 1, \ldots, m$ we let

$$A^i = [A^i_1, A^i_2, \ldots, A^i_n] \in \mathbb{F}_n;$$

thus $A^i$ is the $i$-th row of $A$.

For each $j = 1, \ldots, n$ we let

$$A_j = \begin{bmatrix} A^1_j \\ A^2_j \\ \vdots \\ A^n_j \end{bmatrix} \in \mathbb{F}^m;$$

thus $A_j$ is the $j$-th column of $A$.

**Proposition.** Suppose $m$ and $n$ are positive integers and $A \in \mathbb{F}_n^m$. Then

$$A^i = E^i A = \sum_{k=1}^m A^i_k E^k, \quad A_j = AE_j = \sum_{l=1}^n A^j_l E_l \quad \text{and} \quad A^i_j = E^i (AE_j) = (E_i A) E_j$$

whenever $E^i$ is the $i$-th standard basis covector in $\mathbb{F}_m$ and $E^j$ is the $j$-th standard basis covector in $\mathbb{F}^n$.

**Proof.** Suppose $1 \leq i \leq m$ and $1 \leq j \leq n$. Whenever $l = 1, \ldots, n$ we have

$$(AE_j)_l^k = \sum_{l=1}^m A^i_l (E^j)_l^k = \sum_{l=1}^m A^i_l \delta^i_k = A^k_j = (A_j)^k.$$
Corollary. Suppose \( n \) is a positive integer. Then

\[
X = \sum_{j=1}^{n} X^j E_j \quad \text{whenever} \quad X \in \mathbb{F}^n \quad \text{and} \quad j = 1, \ldots, n
\]

and

\[
Y = \sum_{i=1}^{m} Y^i E^i \quad \text{whenever} \quad Y \in \mathbb{F}^n \quad \text{and} \quad i = 1, \ldots, m.
\]

Proof. Apply the preceding Proposition with \( m = n \) and \( n = 1 \) and with \( m = n \) and \( n = 1 \), respectively. \( \square \)

Corollary. Suppose \( l, m, n \) are positive integers, \( A \in \mathbb{F}^l_m \) and \( B \in \mathbb{F}^m_n \). Then

\[
(AB)^j = AB_j, \quad j = 1, \ldots, n
\]

and

\[
(AB)^i = A^i B, \quad i = 1, \ldots, l.
\]

Proof.