**Fundamental Theorem of Linear Algebra.** Suppose $V$ is a vector space, $S$ and $T$ are finite subsets of $V$, $S$ is independent and $S \subseteq \text{span } T$.

Then $|S| \leq |T|$.

**Remark.** The proof should remind you of row reduction.

**Proof.** We induct on $|T|$. If $|T| = 0$ then $T$ is the empty set, $\text{span } T = \{0\}$ and $S = \emptyset$ since $S$ is independent. Thus $|S| = 0 \leq 0 = |T|$ in this case.

Now suppose, inductively, $n$ is a nonnegative integer and the Theorem holds when $|T| = n$. Let $s_1, \ldots, s_m$ be the distinct members of $S$ and let $t_1, \ldots, t_n, t_{n+1}$ be the distinct members of $T$. We need to show $m \leq n + 1$. In case $m = 1$ this is obviously the case so let us assume henceforth that $m \geq 2$. Let $T' = T \sim \{t_{n+1}\}$.

Since $S \subseteq \text{span } T$, there are scalars

$$c_{i,j}, \ i = 1, \ldots, m, \ j = 1, \ldots, n+1,$$

such that

$$s_i = \sum_{j=1}^{n+1} c_{i,j} t_j, \ i = 1, \ldots, m.$$

**Case One.** $c_{i,n+1} = 0, \ i = 1, \ldots, m$. Then

$$\{s_1, \ldots, s_m\} \subseteq \text{span } T'$$

and $|T'| = n$ so, by the inductive hypothesis, $|S| \leq n < n + 1 = |T|$ and the Theorem, in this case, holds.

**Case Two.** There is $i \in \{1, \ldots, m\}$ such that $c_{i,n+1} \neq 0$. Reindexing if necessary, we may assume

$$c_{1,n+1} \neq 0.$$

Let

$$s'_i = s_i - \frac{c_{i,n+1}}{c_{1,n+1}} s_1, \ i = 2, \ldots, m,$$

let

$$S' = \{s'_2, \ldots, s'_m\}$$

and let

$$d_{i,j} = c_{i,j} - \frac{c_{i,n+1}}{c_{1,n+1}} c_{1,j}, \ i = 2, \ldots, m, \ j = 1, \ldots, n.$$

We leave it as a Homework Problem for the reader to verify that $|S'| = m - 1$ and that $S'$ is independent. We have

$$s'_i = s_i - \frac{c_{i,n+1}}{c_{1,n+1}} s_1 = \sum_{j=1}^{n+1} c_{i,j} t_j - \frac{c_{i,n+1}}{c_{1,n+1}} \sum_{j=1}^{n+1} c_{1,j} t_j = \sum_{j=1}^{n} d_{i,j} t_j$$

whenever $i = 1, \ldots, m$. Thus

$$S' \subseteq \text{span } T'.$$

By the inductive hypothesis,

$$m - 1 = |S'| \leq |T'| = n$$

so $|S| = m \leq n + 1 = |T|$, as desired. ☐