This test consists of five questions, worth a total of 70 points. **Show all your work.**

**Perform elementary row operations by hand. Indicate each operation by mathematical notation** \( (R_i \leftrightarrow R_j, kR_i, R_i + kR_j) \) Unsupported answers may receive no credit.

I. Let

\[
A = \begin{bmatrix}
1 & -1 & -1 & 2 \\
2 & -1 & 1 & 3 \\
-3 & 1 & -3 & -4
\end{bmatrix}
\quad \text{and} \quad
\vec{b} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

1. (10 points) Determine the conditions under which the system of equations \( Ax = \vec{b} \) is consistent.

2. (6 points) If the conditions in part 1 are satisfied, find the general solution (i.e. the set of all the solutions) of the system \( Ax = \vec{b} \).

3. (3 points) Express your answer in part 2 in the form of a linear combination of vectors.

4. (5 points) Let \( \vec{v}_i \) be the ith column of matrix \( A \) where \( i = 1, 2, 3 \). Are \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}_3 \) linearly independent? Justify your answer briefly.
Solution of I continues...
II. Let

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \]

1. (10 points) Find \( A^{-1} \) and solve the system of equations \( A\vec{x} = \vec{b} \).
2. (5 points) Find the adjoint of \( A \), \( \text{adj}(A) \).
3. (5 points) Let \( \vec{v}_i \) be the ith column of matrix \( A \) where \( i = 1, 2, 3 \). Are \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}_3 \) linearly independent? Justify your answer briefly.
III. (14 points) Let \( A = [A_1, A_2, A_3, A_4] \) be a 4 \( \times \) 4 matrix and \( \det(A) = 10 \). Compute each of the following:

1. \( \det(B) \) where \( B = [5A_1, 5A_2, 5A_3, 5A_4] \);
2. \( \det(C) \) where \( C = [A_1 + 5A_4, A_3, A_2, 8(A_1 + 5A_4)] \);
3. \( \det(D) \) where \( D = [A_1 + 5A_4, A_3, A_2, A_4] \);
4. \( \det(E) \) where \( E = [A_1 + 5A_4, -2A_3, A_2, 7A_3 + A_4] \).
5. \( x_3 \), the (3, 1) entry of \( \vec{x} \) which is the solution of the system of equations \( A\vec{x} = \vec{b} \) where \( \vec{b} = A_3 - 6A_1 \).
IV. (6 points) Let \( A = I - a v v^T \), where \( v \) is a nonzero \( n \times 1 \) vector in \( \mathbb{R}^n \), \( a \) is a nonzero number and \( I \) is the \( n \times n \) identity matrix. Let \( a = \frac{2}{(v^T v)} \). Compute \((I - a v v^T)^2\) and simplify your answer.

V. (6 points) True-False. Write T or F in the blank to the left of each statement.

1. An \( m \times n \) system of linear equations always has a unique solution if \( m = n \).

T

2. An \( m \times n \) system of linear equations is always inconsistent if \( m > n \).

F

3. \( \det(A + B) = \det(A) + \det(B) \) where \( A \) and \( B \) are two \( n \times n \) matrices.

T

4. \( (A + B)^{-1} = A^{-1} + B^{-1} \) where \( A \) and \( B \) are two \( n \times n \) invertible matrices.

F

5. \( (A + B)^T = A^T + B^T \) where \( A \) and \( B \) are two \( m \times n \) matrices.

T

6. Let \( a, b, \cdots, j \) be ten arbitrary numbers. \( A^3 = O \) if

\[
A = \begin{bmatrix}
0 & 0 & a & b & c & d \\
0 & 0 & 0 & e & f & g \\
0 & 0 & 0 & h & i \\
0 & 0 & 0 & 0 & j \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]