1. Let
   \[ B = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0, \ 1 < x^2 - y^2 < 4 \text{ and } 4 < x^2 + y^2 < 9 \} \]
   and let
   \[ F(x, y) = (x^2 - y^2, x^2 + y^2) \quad \text{for } (x, y) \in B. \]
   I tell you that \( F \) carries \( A \) in one-to-one fashion onto the rectangle
   \[ \{(u, v) \in \mathbb{R}^2 : 1 < u < 4 \text{ and } 4 < v < 9 \}. \]

   (a) **10 pts.** Use the Change of Variables formula to calculate
   \[ \int \int_B x^2 + y^2 \, dxdy. \]

   (b) **10 pts.** Calculate the inverse of \( F \).

**Solution.** First note that
   \[ J_F(x, y) = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 8xy \neq 0 \quad \text{for } (x, y) \in B. \]
Let $A$ be the rectangle
\[ \{(u, v) \in \mathbb{R}^2 : 1 < u < 4 \text{ and } 4 < v < 9\} . \]

Then $F^{-1}$ carries $A$ in one-to-one fashion onto $B$ since $F$ carries $B$ in one-to-one fashion onto $A$. Moreover, by the Chain Rule and the product rule for determinants (see (9) on page 1005) we have
\[ J_{F^{-1}}(u, v) = \frac{1}{J_F(F^{-1}(u, v))} \quad \text{for } (u, v) \in A . \]

By the Change of Variable Formula for Multiple Integrals (Theorem 1 on page 1004) we have
\[
\int \int_B f(x, y) \, dx \, dy = \int \int_{F^{-1}[A]} f(x, y) \, dx \, dy
\]
\[
= \int \int_A f(F^{-1}(u, v)) J_{F^{-1}}(u, v) \, dudv
\]
\[
= \int \int_A \frac{f(F^{-1}(u, v))}{J_F(F^{-1}(u, v))} \, dudv .
\]

(1)

In the original statement of the problem I should have had you integrate $f(x, y) = xy$ instead of $f(x, y) = x^2 + y^2$ because this leads to an easy integral as follows. If $f(x, y) = xy$ for $(x, y) \in B$ we have
\[
\frac{f(x, y)}{J_F(x, y)} = \frac{xy}{8xy} = \frac{1}{8}
\]
so
\[
\int \int_B xy \, dx \, dy = \int \int_A \frac{1}{8} \, dudv = \frac{(4 - 1)(9 - 4)}{8} = \frac{15}{8} .
\]

Back to the problem as stated. Now let $f(x, y) = x^2 + y^2$ for $(x, y) \in B$. Fix $(x, y) \in B$ and let
\[ (u, v) = F(x, y) = (x^2 - y^2, x^2 + y^2) . \]

It follows that
\[ F^{-1}(u, v) = (x, y) . \]

Now
\[ x^2 y^2 = (x^2 + y^2) - (x^2 - y^2) = v - u \frac{4}{4} \]
so
\[ xy = \frac{\sqrt{v - u}}{2} \quad \text{since } v > u. \]

Consequently,
\[
\frac{f(F^{-1}(u, v))}{J_F(F^{-1}(u, v))} = \frac{v}{4\sqrt{v - u}}
\]
which gives

\[
\int \int_B x^2 + y^2 \, dxdy
\]

\[
= \int \int_A \frac{v}{4\sqrt{v-u}} \, dudv
\]

\[
= \int_1^4 \left( \int_1^9 \frac{v}{4\sqrt{v-u}} \, dv \right) \, du
\]

\[
= \frac{464}{15} \sqrt{2} - \frac{14}{5} \sqrt{3} - \frac{35}{3} \sqrt{5}
\]

(I meant to have you end up with a real easy integral. So I should have told you to integrate \(xy\) instead of \(x^2 + y^2\).)

To calculate the inverse of \(F\) we need to solve \(u = x^2 - y^2\) and \(v = x^2 + y^2\) for \(u\) and \(v\). *Keep in mind that \(v > u\) for \((x, y) \in B\).* Adding these equations we get \(u + v = 2x^2\) so \(x = \sqrt{(u+v)/2}\) and subtracting the second from the first we get \(v - u = 2y^2\) so \(y = \sqrt{(v-u)/2}\). That is,

\[
F^{-1}(u, v) = \left( \sqrt{\frac{u+v}{2}}, \sqrt{\frac{v-u}{2}} \right) \quad \text{for} \quad (u, v) \in A.
\]

Note that having these formulae will allow you to calculate \(J_{F^{-1}}(u, v)\) directly, giving another way to do (a).

2. Let

\[
P(x, y) = 2x^2 + x \quad \text{and} \quad Q(x, y) = -3x^2 + y \quad \text{for} \quad (x, y) \in \mathbb{R}^2
\]

and let \(R\) be the triangle in \(\mathbb{R}^2\) with vertices \((0, 0), (1, 1), (0, 1)\).

(a) 5 pts. Calculate

\[
\int \int_R x \, dxdy.
\]

(b) 5 pts. Calculate

\[
\int_C P \, dx + Q \, dy
\]

where \(C\) is the boundary of \(R\) traversed in the counterclockwise sense.

(c) 5 pts. Explain how the answer to either one of (a) or (b) may be used to find the answer to the other.

**Solution.** We have

\[
\int \int_R x \, dxdy = \int_0^1 \left( \int_0^x x \, dx \right) \, dy = \frac{1}{6}.
\]
Moreover, if $C_i$, $i = 1, 2, 3$, are the segments joining $(0, 0)$ to $(1, 1)$; $(1, 1)$ to $(0, 1)$; and $(0, 1)$ to $(0, 0)$, respectively, we have

\[
\int_C P\, dx + Q\, dy = \left( \int_{C_1} + \int_{C_2} + \int_{C_3} \right) P\, dx + Q\, dy
\]

\[
= \int_0^1 (2t^2 + t)\, dt + (-3t^2 + t)\, dt
\]

\[
- \int_0^1 (2t^2 + t)\, dt + (-3t^2 + 1)\, dt
\]

\[
- \int_0^1 (20^2 + 0)\, dt + (-30^2 + t)\, dt = -1.
\]

Now Green’s Theorem says

\[
\int_C P\, dx + Q\, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy = \int \int_R -6x\, dxdy
\]

so the answer to (b) should be $-6$ times the answer to (a).

3. 10 pts. Let

\[
\mathbf{F}(x, y) = (y, 0) \quad \text{for } (x, y) \in \mathbb{R}^2
\]

and let $C$ be the curve in $\mathbb{R}^3$ which consists of the line segment which goes from $(1, 0)$ to $(2, 0)$ and then follows the circle with center $(0, 0)$ and radius 2 from $(2, 0)$ counterclockwise to $(0, 2)$. Calculate

\[
\int_C \mathbf{F} \cdot \mathbf{T}\, ds.
\]

**Solution.** Let $C_1$ and $C_2$ be the line segment which goes from $(1, 0)$ to $(2, 0)$ and let $C_2$ be the part of the circle with center $(0, 0)$ and radius 2 from $(2, 0)$ counterclockwise to $(0, 2)$. Then

\[
\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^1 (0, 0) \cdot (1, 0)\, dt = 0
\]

and

\[
\int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{\pi/2} (2\sin t, 0) \cdot (-2\sin t, 2\cos t)\, dt = \int_0^{\pi/2} -4\sin^2 t\, dt = -\pi.
\]

So

\[
\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds = 0 + (-\pi) = -\pi.
\]

4. Let

\[
\mathbf{F}(x, y) = (x + y^2, 2xy) \quad \text{for } (x, y) \in \mathbb{R}^2.
\]
(a) **5 pts.** Find a continuously differentiable function $f$ on $\mathbb{R}^2$ such that $\mathbf{F} = \nabla f$.

(b) **5 pts.** Calculate
\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds
\]
where $C$ is a curve which goes from $(1, 1)$ to $(-2, -3)$.

**Solution.** By partial integration we have
\[
f(x, y) = A(y) + \frac{x^2}{2} + xy^2 \quad \text{and} \quad f(x, y) = B(x) + xy^2
\]
so we can take $B(x) = x^2/2$ and $A(y) = 0$ and let
\[
f(x, y) = \frac{x^2}{2} + xy^2.
\]
Because $\mathbf{F} = \nabla f$ we have
\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds = f(-2, -3) - f(1, 1) = -\frac{35}{2}.
\]

5. **10 pts.** Let
\[
S = \{(x, y, z) : z = xy \quad \text{and} \quad x^2 + y^2 \leq 1\}
\]
and let
\[
\mathbf{F}(x, y, z) = (y, x, 0) \quad \text{for} \quad (x, y, z) \in \mathbb{R}^3.
\]
Calculate the flux
\[
\int \int_S \mathbf{F} \cdot \mathbf{n} \, dS
\]
where $\mathbf{n}$ is the upward pointing unit normal to $S$.

**Solution.** Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and note that $S$ is the graph of $f(x, y) = xy$ over $D$. Thus as
\[
\mathbf{n} \, dS = (-f_x, -f_y, 1) \, dx\,dy = (-y, -x, 1)
\]
we obtain
\[
\int \int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int \int_D (y, x, 0) \cdot (-y, -x, 1) \, dx\,dy
\]
\[
= \int \int_D -(x^2 + y^2) \, dx\,dy
\]
\[
= -\int_0^1 \left( \int_0^{2\pi} (r^2) r \, dr \right) \, d\theta
\]
\[
= -\frac{\pi}{2}.
\]

6. **10 pts.** Let
\[
\mathbf{F}(x, y, z) = (yz, xz, xy) \quad \text{for} \quad (x, y, z) \in \mathbb{R}^3.
\]
Find a continuously differentiable function $f$ on $\mathbb{R}^3$ such that $\mathbf{F} = \nabla f$. 


Solution. By partial integration we obtain
\[ f(x, y, z) = A(y, z) + xyz, \quad f(x, y, z) = B(x, z) + xy, \quad f(x, y, z) = C(x, y) + xyz. \]
Letting \( A(y, z) = 0, \ B(x, z) = 0 \) and \( C(x, y) = 0 \) we obtain
\[ f(x, y, z) = xyz \quad \text{for } (x, y, z) \in \mathbb{R}^3. \]

7. Let
\[ \mathbf{F}(x, y, z) = (x + e^{yz}, y + \sin xz, z + \cos xy) \quad \text{for } (x, y, z) \in \mathbb{R}^3. \]
(a) 5 pts. Calculate the divergence of \( \mathbf{F} \).
(b) 10 pts. Use the Divergence Theorem to evaluate
\[ \int \int S \mathbf{F} \cdot \mathbf{n} \, ds \]
where \( S \) is the surface which bounds the region
\[ T = \{ (x, y, z) : z \geq 0 \text{ and } z^2 \leq 25 - x^2 - y^2 \} \]
and where \( \mathbf{n} \) is the unit normal to \( S \) which points out of \( T \). You only need express your answer as iterated single integrals.

Solution. We have
\[ \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (x + e^{yz}) + \frac{\partial}{\partial y} (y + \sin xz) + \frac{\partial}{\partial z} (z + \cos xy) = 1 + 1 + 1 = 3. \]
Let \( D = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25 \} \) and note that \( T \) is the region whose projection onto the \( xy \)-plane is \( D \) and which is between \( z = 0 \) and \( z = \sqrt{25 - x^2 - y^2} \). By the Divergence Theorem,
\[ \int \int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int \int_T \nabla \cdot \mathbf{F} \, dV = 3 \text{Volume}(T) \]
\[ = \int_D \sqrt{25 - x^2 - y^2} \, dxdy \]
\[ = \int_0^5 \left( \int_0^{2\pi} \sqrt{25 - r^2} \, dr \right) d\theta \]
\[ = \frac{250\pi}{3}. \]

8. Let
\[ \mathbf{F}(x, y, z) = (y^2, z^2, x^2) \quad \text{for } (x, y, z) \in \mathbb{R}^3. \]
(a) 5 pts. Calculate the curl of \( \mathbf{F} \).
(b) 10 pts. Use Stokes’s Theorem to evaluate
\[ \int_C \mathbf{F} \cdot \mathbf{T} \, ds \]
where \( C \) is the intersection of the cylinder \( x^2 + y^2 = 2y \) with the plane \( z = y \) oriented counterclockwise when viewed from above. You only need to express your answer as iterated single integrals.

Solution. We have
\[ (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2(x, y, z). \]

Our surface \( S \) is the graph of \( f(x, y) = y \) over the disk \( D \) in \( \mathbb{R}^2 \) with center \((0, 1)\) and radius 1. Moreover,
\[ D = \{(r \cos \theta, r \sin \theta) : 0 \leq \theta \leq \pi \text{ and } 0 \leq r \leq 2 \sin \theta\}. \]

Thus
\[ \mathbf{n} \, dS = (-f_x, -f_y, 1) \, dxdy = (0, -1, 1). \]

Thus
\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\]
\[ = \int_D -2(z, x, y) \cdot (0, -1, 1) \, dxdy
\]
\[ = 2 \int_D y - x \, dxdy
\]
\[ = 2 \int_0^\pi \left( \int_0^{2 \sin \theta} (r \sin \theta - r \cos \theta) r \, dr \right) d\theta
\]
\[ = 2\pi. \]