I have neither given nor received aid in the completion of this test.
Signature:

TO GET FULL CREDIT YOU MUST SHOW ALL WORK!

<table>
<thead>
<tr>
<th></th>
<th>Your Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5 pts.</td>
</tr>
<tr>
<td>2</td>
<td>20 pts.</td>
</tr>
<tr>
<td>3</td>
<td>5 pts.</td>
</tr>
<tr>
<td>4</td>
<td>5 pts.</td>
</tr>
<tr>
<td>5</td>
<td>10 pts.</td>
</tr>
<tr>
<td>6</td>
<td>15 pts.</td>
</tr>
<tr>
<td>7</td>
<td>10 pts.</td>
</tr>
<tr>
<td>8</td>
<td>5 pts.</td>
</tr>
<tr>
<td>9</td>
<td>5 pts.</td>
</tr>
<tr>
<td>10</td>
<td>10 pts.</td>
</tr>
<tr>
<td>11</td>
<td>15 pts.</td>
</tr>
<tr>
<td></td>
<td>Total 105 pts.</td>
</tr>
</tbody>
</table>

The average on this 50 minute test was 54.16 and the standard deviation was 17.25.

1. 5 pts. Let $P$ be the plane consisting of those $(x, y, z)$ such that $x + y + z = 1$. Exhibit parametric equations for the line passing through $(1, 2, 3)$ which meets $P$ in a right angle.

Solution. Let $\mathbf{n} = (1, 1, 1)$; then $\mathbf{n}$ is normal to $P$ so
$$\mathbf{r}(t) = (1, 2, 3) + t(1, 1, 1), \quad t \in \mathbb{R},$$
parameterizes the line.

2. (a) 5 pts. Show that the points $(1, 1, 0), (1, 0, 0), (0, 1, 1)$ do not lie on a line.

Solution. Let $\mathbf{v} = (1, 0, 0) - (1, 1, 0) = (0, -1, 0)$ and let $\mathbf{w} = (0, 1, 1) - (1, 1, 0) = (-1, 0, 1)$. Then
$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} i & j & k \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = -i - k = (-1, 0, -1).$$
Since $v \times w \neq 0$ the points do not lie on a line.

(b) 5 pts. Let $P$ be the plane containing the points in part (a). Exhibit scalars $a, b, c, d$ such the point $(x, y, z)$ is in $P$ if and only if

$$ax + by + cz = d.$$ 

**Solution.** $n = v \times w$ is normal to $P$ and $(1, 1, 0)$ lies in $P$. So

$$-x - z = (x, y, z) \cdot (-1, 0, -1) = (1, 1, 0) \cdot (-1, 0, -1) = -1$$

is an equation for $P$ so we may take $a = -1, b = 0, c = -1, d = -1$.

(c) 10 pts. Let $q = (1, 2, 3)$. Show that $q$ does not lie on $P$ and determine the point $p$ in $P$ which is closest to $q$.

**Solution.** $-1 = q \cdot n = (1, 2, 3) \cdot (-1, 0, -1) = -1$ so $q$ does not lie in $P$. Moreover,

$$r(t) = q + t n = (1, 2, 3) + t(-1, 0, -1), \quad t \in \mathbb{R}$$

parameterizes the line passing through $q$ perpendicular to $P$ so $p = r(t)$ if

$$-1 = r(t) \cdot n = ((1, 2, 3) + t(-1, 0, -1)) \cdot (-1, 0, -1) = -4 + 2t$$

so $t = 3/2$ and

$$p = r \left( \frac{3}{2} \right) = (1, 2, 3) + \frac{3}{2}(-1, 0, -1) = \frac{1}{2} (-1, 4, 3).$$

3. 5 pts. Suppose for $P_i$ is the plane with equation

$$a_ix + b_iy + c_iz = d_i$$

for each $i = 1, 2$. How do you tell if $P_1$ is parallel to $P_2$?

**Solution.** Let $n_i = (a_i, b_i, c_i)$ for $i = 1, 2$. Then $n_i$ is normal to $P_i$, $i = 1, 2$, so $P_1 \parallel P_2$ if and only if $n_1 \times n_2 = 0$.

4. 5 pts. Compute $\text{comp}_{(3,0,4)}(1,2,3)$.

**Solution.** We have

$$\text{comp}_{(3,0,4)}(1,2,3) = \frac{(1,2,3) \cdot (3,0,4)}{|(3,0,4)|} = \frac{1 \cdot 3 + 2 \cdot 0 + 3 \cdot 4}{\sqrt{3^2 + 0^2 + 4^2}} = \frac{3}{5} = 1.5.$$ 

5. 10 pts. Suppose $I$ is an open interval and $r : I \to \mathbb{R}^3$ is a twice continuously differentiable curve in $\mathbb{R}^3$. Suppose $t_0 \in I$ and

$$r'(t_0) = (1, 0, 1), \quad r''(t_0) = (1, 1, 1).$$

Determine $T(t_0), |v'(t_0)|, N(t_0),$ and $\kappa(t_0)$.

**Solution.** We have $v(t_0) = r'(t_0) = (1, 0, 1)$ so $|v'(t_0)| = \sqrt{2}$.
and 
\[ T(t_0) = \frac{v}{|v|}(t_0) = \frac{1}{\sqrt{2}}(1, 0, 1). \]
We have
\[ a(t_0) = r''(t_0) = (1, 1, 1) \]
and
\[ |v'(t_0) = (a \cdot T)(t_0) = \frac{2}{\sqrt{2}} = \sqrt{2}. \]
We have
\[ (\kappa|v|^2N)(t_0) = (a - (a \cdot T)T)(t_0) = (1, 1, 1) - \sqrt{2} \frac{1}{\sqrt{2}} (1, 0, 1) = (0, 1, 0). \]
Since \( \kappa(t_0) > 0 \) we find that
\[ N(t_0) = (0, 1, 0) \quad \text{and} \quad \kappa(t_0) = \frac{1}{|v|^2(t_0)} = \frac{1}{2}. \]

6. 15 pts. Let \( f(x, y) = xy - x + 2 \) and let \( R = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 4 - x^2 \} \).
Find the maxima and minima of \( f \) on \( R \). (Note that \( f \) is continuous and \( R \) is closed and bounded so both maxima and minima exist.)

**Solution.** First we find the critical points. We have
\[ \frac{\partial f}{\partial x} = y - 1 \quad \text{and} \quad \frac{\partial f}{\partial y} = x \]
so the unique critical point is \((0, 1)\) which lies in \( R \).

Next we note the boundary of \( R \) consists of the arcs which are the ranges of the curves
\[ C_1(x) = (x, 0) \quad \text{for } -2 < x < 2 \]
and
\[ C_2(x) = (x, 4 - x^2) \quad \text{for } -2 < x < 2 \]
together with the endpoints of these arcs, namely the points \((\pm 2, 0)\). Since
\[ \frac{d}{dx} f(C_1(x)) = \frac{d}{dx} (-x + 2) = -1 \]
there are no candidates for minimum or maximum points of \( f \) on \( R \) on \( C_1 \). Since
\[ \frac{d}{dx} f(C_2(x)) = \frac{d}{dx} (x(4 - x^2) - x + 2) = 3(1 - x^2) \]
we find that \((\pm 1, 3)\) are candidates for minimum or maximum points of \( f \) on \( R \) on \( C_2 \). Finally, we consider the endpoints \((\pm 2, 0)\).

We consider the table

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( f(x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>
Thus the minimum value of $f$ on $R$ is 0 which occurs at $(-1, 3)$ and $(2, 0)$ and the maximum value of $f$ on $R$ is 4 which occurs at $(1, 3)$ and $(-2, 0)$.

7. 10 pts. Let

$$f(x, y) = \begin{cases} \frac{-x}{x^2 + y^2} & \text{if } x < y, \\ \frac{y}{x^2 + y^2} & \text{if } x \geq y. \end{cases}$$

Does $\lim_{(x,y) \to (0,0)} f(x, y)$ exist? If so, what is it? Why?

Solution. If $0 < x < \infty$ we have $f(x, 0) = 0$ and $f(x, x) = 1/x$. Thus

$$\lim_{x \to 0} f(x, 0) = 0 \quad \text{and} \quad \lim_{x \to 0} f(x, x) = \infty$$

so the limit does not exist.

8. 5 pts. Exhibit an equation for the tangent plane to the graph of $z = \cos xy$ at $(1, \pi/2, 0)$.

Solution. Note that $f(1, \pi/2) = 0$; otherwise the problem is incorrectly posed. We have

$$\frac{\partial f}{\partial x} = -y \sin xy \quad \text{which at } (1, \pi/2) \text{ equals } -\frac{\pi}{2}$$

and

$$\frac{\partial f}{\partial y} = -x \sin xy \quad \text{which at } (1, \pi/2) \text{ equals } -1.$$

Thus

$$z = -\frac{\pi}{2}(x - 1) + (-1) \left(y - \frac{\pi}{2}\right) = -\frac{\pi}{2}x - y + \pi$$

is the desired equation.

9. 5 pts. Calculate

$$\frac{\partial^2}{\partial x \partial y} e^{xyz}.$$

Solution. We have

$$\frac{\partial^2}{\partial x \partial y} e^{xyz} = \frac{\partial}{\partial x} (xze^{xyz})$$

$$= xe^{xyz} + (xz)(yz)e^{xyz}$$

$$= z(1 + xyz)e^{xyz}.$$

9. 11 pts. Suppose $g, h$ are continuously differentiable functions on the interval $I$, $f$ is a continuously differentiable function on $\mathbb{R}^2$ and

$$w(t) = f(g(t), h(t)) \quad \text{for } t \in I.$$

Suppose

$$2 \in I, \quad g(2) = 1, \quad g'(2) = 1, \quad h(2) = 4, \quad h'(2) = 3$$

as well as

$$f(1, 4) = 5, \quad \frac{\partial f}{\partial x}(1, 4) = 3$$

and

$$w'(2) = 4.$$
Determine \( \frac{\partial f}{\partial y}(1,4) \).

Solution. The Chain Rule says that
\[
 w'(t) = \frac{\partial f}{\partial x}(g(t),h(t))g'(t) + \frac{\partial f}{\partial y}(g(t),h(t))h'(t)
\]
for any \( t \in I \). If \( t = 2 \) then \((g(t),h(t)) = (1,4)\) so
\[
 4 = \frac{\partial f}{\partial x}(1,4)(1) + \frac{\partial f}{\partial y}(1,4)(3) = (3)(1) + \frac{\partial f}{\partial y}(1,4)(3)
\]
so
\[
 \frac{\partial f}{\partial y}(1,4) = \frac{1}{3}(4-3) = \frac{1}{3}.
\]

11. 15 pts. Suppose \( a, b, c, d \) are points in \( \mathbb{R}^3 \) which do not lie in a plane. Let \( P \) be the plane passing through \( a, b, c \) and let \( Q \) be the plane passing through the midpoints of the segments joining \( a \) to \( d \), \( b \) to \( d \) and \( c \) to \( d \), respectively. Show that \( P \) and \( Q \) are parallel.

Solution. The vector
\[
(b - a) \times (c - a)
\]
is normal to \( P \). The vectors
\[
m_a = \frac{1}{2}(d - a), \quad m_b = \frac{1}{2}(d - b), \quad m_c = \frac{1}{2}(d - c)
\]
are the midpoints of the segments joining \( a \) to \( d \), \( b \) to \( d \) and \( c \) to \( d \), respectively. We have
\[
(m_b - m_a) \times (m_c - m_a) = \left( \frac{1}{2}(a - b) \right) \times \left( \frac{1}{2}(a - c) \right) = \frac{1}{4}((b - a) \times (c - a))
\]
from which we infer that \( P \) and \( Q \) are parallel.