1. The whole story for two lines in \( \mathbb{R}^3 \).

Suppose \( \mathbf{a}, \mathbf{b}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \) and \( \mathbf{v} \) and \( \mathbf{w} \) are nonzero. Let
\[
\mathbf{r}(t) = \mathbf{a} + t \mathbf{v} \quad \text{for } t \in \mathbb{R}
\]
and let
\[
\mathbf{s}(t) = \mathbf{b} + t \mathbf{w} \quad \text{for } u \in \mathbb{R}.
\]
Let \( L \) and \( M \) be the ranges of \( \mathbf{r} \) and \( \mathbf{s} \), respectively. That is,
\[
L = \{ \mathbf{r}(t) : t \in \mathbb{R} \} \quad \text{and} \quad M = \{ \mathbf{s}(u) : u \in \mathbb{R} \}.
\]
We have \( L \parallel M \) if and only if \( \mathbf{v} \parallel \mathbf{w} \) if and only if \( \mathbf{v} \times \mathbf{w} = \mathbf{0} \). If this is so then, as I hope is clear to you, \( L = M \) if and only if \( \mathbf{v} \times \mathbf{w} = \mathbf{0} \). If this is the case then, as I hope is clear to you, \( L = M \) if and only if \( \mathbf{v} \parallel \mathbf{w} \) if and only if \( \mathbf{v} \times \mathbf{w} = \mathbf{0} \).

So suppose \( \mathbf{v} \times \mathbf{w} \neq \mathbf{0} \). Let
\[
\mathbf{n} = \mathbf{v} \times \mathbf{w}, \quad c = \mathbf{a} \cdot \mathbf{n}, \quad d = \mathbf{w} \cdot \mathbf{n}.
\]
Let
\[
P = \{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{n} = c \} \quad \text{and let} \quad Q = \{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{n} = d \}.
\]
Evidently
\[
L \subset P, \quad M \subset Q, \quad P \parallel Q \quad \text{and} \quad P = Q \quad \text{if and only if} \quad c = d.
\]

Let find \( t, u, v \) such that
\[
(1) \quad \mathbf{a} + t \mathbf{v} = \mathbf{b} + u \mathbf{w} + v \mathbf{n}.
\]
It is geometrically clear that \( t, u, v \) exist and are unique. Dotting (1) with \( \mathbf{n} \) we find that
\[
v = \frac{c - d}{|\mathbf{n}|^2}.
\]
Dotting (1) with \( \mathbf{v} \) and then with \( \mathbf{w} \) and transposing a little we find that
\[
(2) \quad t|\mathbf{v}|^2 - u \mathbf{v} \cdot \mathbf{w} = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{v},
\]
\[
- t \mathbf{v} \cdot \mathbf{w} + u |\mathbf{w}|^2 = (\mathbf{a} - \mathbf{b}) \cdot \mathbf{w}.
\]
which is equivalent to the matrix equation
\[
(3) \quad \begin{bmatrix}
|\mathbf{v}|^2 & - \mathbf{v} \cdot \mathbf{w} \\
- \mathbf{v} \cdot \mathbf{w} & |\mathbf{w}|^2
\end{bmatrix}
\begin{bmatrix}
t \\
u
\end{bmatrix}
= \begin{bmatrix}
(\mathbf{b} - \mathbf{a}) \cdot \mathbf{v} \\
(\mathbf{a} - \mathbf{b}) \cdot \mathbf{w}
\end{bmatrix}
\]
the unique solution of which, by Cramer’s Rule, is
\[
t = \begin{vmatrix}
|\mathbf{v}|^2 & - \mathbf{v} \cdot \mathbf{w} \\
- \mathbf{v} \cdot \mathbf{w} & |\mathbf{w}|^2
\end{vmatrix}, \quad u = \begin{vmatrix}
|\mathbf{v}|^2 & (\mathbf{b} - \mathbf{a}) \cdot \mathbf{v} \\
- \mathbf{v} \cdot \mathbf{w} & (\mathbf{a} - \mathbf{b}) \cdot \mathbf{w}
\end{vmatrix}.
\]
This gives
\[
t = \frac{\mathbf{b} - \mathbf{a} \cdot \text{comp}_w \mathbf{v}}{|\mathbf{w}|^2 |\mathbf{v} \times \mathbf{w}|} \quad \text{and} \quad u = \frac{(1 - b) \cdot \text{comp}_w \mathbf{v}}{|\mathbf{v} \times \mathbf{w}|}.
\]
Neat, huh? But check my calculations!