Math 103.02 Quiz Three

I have neither given nor received aid in the completion of this test.
Signature:

Let L be the line in $\mathbb{R}^3$ with parameterization

$$\mathbf{r}(t) =< 1,1,2 > +t < 1,0,1 > \quad \text{for } t \in \mathbb{R}$$

and let M be the line in $\mathbb{R}^3$ with parameterization

$$\mathbf{s}(u) =< 2,2,1 > +u < 1,1,1 > \quad \text{for } u \in \mathbb{R}.$$

(1.) Show that L and M are not parallel.

Since L and M are not parallel there are unique parallel planes P and Q such that $L \subset P$ and $M \subset Q$.

(2.) Find equations for the planes P and Q and show that $P \cap Q = \emptyset$. Find the distance between P and Q.

(3.) It follows from (2.) that $L \cap M = \emptyset$. Find $t, u \in \mathbb{R}$ such that if N is the line containing $\mathbf{r}(t)$ and $\mathbf{s}(u)$ then N is normal to both P and Q.

Solution. Let

$$\mathbf{a} =< 1,1,2 >, \quad \mathbf{v} =< 1,0,1 >, \quad \mathbf{b} =< 2,2,1 >, \quad \mathbf{w} =< 1,1,1 >$$

let

$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{v},\ t \in \mathbb{R} \quad \text{and let} \quad \mathbf{s}(u) = \mathbf{b} + u\mathbf{w},\ u \in \mathbb{R}.$$

Then $\mathbf{r}$ is parameterization of L and $\mathbf{s}$ is a parameterization of M. We have

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} =< -1,0,1 > .$$

Since $\mathbf{n} \neq \mathbf{0}$ we conclude that L and M are not parallel. Let

$$c = \mathbf{a} \bullet \mathbf{n} = 1 \quad \text{and let} \quad d = \mathbf{b} \bullet \mathbf{n} = -1.$$

Then

$$\mathbf{x} \bullet \mathbf{n} = c \quad \text{and} \quad \mathbf{x} \bullet \mathbf{n} = d$$

are equations for P and Q, respectively (Do you see why?). Moreover,

$$\frac{c}{|\mathbf{n}|^2}\mathbf{n} \in P \quad \text{and} \quad \frac{d}{|\mathbf{n}|^2}\mathbf{n} \in Q$$

so the distance between P and Q is

$$\left| \frac{c}{|\mathbf{n}|^2}\mathbf{n} - \frac{d}{||\mathbf{n}|^2}\mathbf{n} \right| = \frac{|c-d|}{|\mathbf{n}|} = \frac{|-2|}{\sqrt{2}} = \sqrt{2}.$$

In the last part we are looking for $t, u \in \mathbb{R}$ such that

$$(\mathbf{r}(t) - \mathbf{s}(u)) \bullet \mathbf{u} = 0 \quad \text{and} \quad (\mathbf{r}(t) - \mathbf{s}(u)) \bullet \mathbf{v} = 0.$$

Now

$$\begin{aligned}\mathbf{r}(t) - \mathbf{s}(u) &=< 1, 1, 2 > +t < 1, 0, 1 > -(< 2, 2, 1 > +u < 1, 1, 1 >) \\ &=< -1 + t - u, -1, -u, 1 + t - u >;\end{aligned}$$

dotting with $\mathbf{v}$ we obtain

$$2t - 2u = 0$$

and dotting with $\mathbf{w}$ we obtain

$$2t - 3u = 1$$

so $t = -1$ and $u = -1$.