Math 103.02 Quiz Ten
Due Monday, November 29.

I have neither given nor received aid in the completion of this test.
Signature:

(1) Let $C$ be the segment joining $(1,1)$ to $(3,5)$. Calculate
\[
\int_C x \, dx \quad \int_C x \, ds \quad \text{and} \quad \int_C (x, y) \cdot T \, ds.
\]

Solution. Let $P(t) = (x(t), y(t)) = (1 + 2t, 1 + 4t)$, $0 \leq t \leq 1$. Then $P$ parameterizes $C$ and $P'(t) = (2, 4)$, $0 \leq t \leq 1$, so
\[
\int_C x \, dx = \int_0^1 (1 + 2t) \, 2 \, dt = 4 \quad \text{since} \quad dx = 2 \, dt;
\]
\[
\int_C x \, ds = \int_0^1 (1 + 2t) \sqrt{2^2 + 4^2} \, dt = 4 \sqrt{5};
\]
\[
\int_C (x, y) \cdot T \, ds = \int_0^1 (1 + 2t, 1 + 4t) \cdot (2, 4) \, dt = 16.
\]

A better way to do the first of these is to recognize that $(x, 0) = \nabla f(x, y)$ where $f(x, y) = x^2/2$ for $(x, y) \in \mathbb{R}^2$ so
\[
\int_C x \, dx = \int \nabla f \cdot T \, dx = f(3, 5) - f(1, 1) = \frac{3^2}{2} - \frac{1^2}{2} = 4.
\]

You cannot do the second one this way. A better way to do the third of these is to recognize that $(x, y) = \nabla f(x, y)$ where $f(x, y) = (x^2 + y^2)/2$ for $(x, y) \in \mathbb{R}^2$ so
\[
\int_C (x, y) \cdot T \, ds = f(3, 5) - f(1, 1) = \frac{3^2 + 5^2}{2} - \frac{1^2 + 1^2}{2} = 16.
\]

(2) Let $C$ be curve joining $(1,0,0)$ to $(-1,0,0)$ which lies in
\[
\left\{(x, y, z) : x^2 + \frac{y^2}{2} + \frac{z^2}{2} = 1, \quad y = z \text{ and } z \geq 0\right\}.
\]

Calculate
\[
\int_C x \, ds \quad \text{and} \quad \int_C (x, 0, z) \cdot T \, ds.
\]

Solution. The projection of this curve on the $xy$-plane equals $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ and } y \geq 0\}$; (note that since $z \geq 0$ and $y = z$ on the curve we have
$y \geq 0$ on the curve. So
\[ (x(t), y(t), z(t)) = (\cos t, \sin t, \sin t), \quad 0 \leq t \leq \pi, \]
is a parameterization of the curve. Thus
\[ \int_C x \, ds = \int_0^\pi \cos t \sqrt{(-\sin t)^2 + (\cos t)^2 + (\cos t)^2} \, dt = 0, \]
which can also be seen by the fact that the curve is symmetric with respect to reflection across the plane $x = 0$.

Finally,
\[ \int_C (x, 0, z) \cdot \mathbf{T} \, ds \]
\[ = \int_0^\pi (\cos t, 0, \sin t) \cdot (-\sin t, \cos t, \cos t) \, dt \]
\[ = \int_0^\pi (\cos t)(-\sin t) + (\sin t)(\cos t) \, dt \]
\[ = \int_0^\pi 0 \, dt = 0. \]