1. The equality of mixed partial derivatives.

**Theorem 1.1.** Suppose $A \subset \mathbb{R}^2$ and 
\[ f : A \to \mathbb{R}. \]
Suppose $(a, b)$ is an interior point of $A$ near which the partial derivatives
\[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \]
exist. Suppose, in addition, that
\[ \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x} \]
exist near $(a, b)$ and are continuous at $(a, b)$. Then
\[ \frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b). \]

**Proof.** Let
\[ S(x, y) = f(x, y) - f(x, b) - f(a, y) + f(a, b) \quad \text{for} \ (x, y) \in A. \]
Let
\[ A(x, y) = f(x, y) - f(a, y) \quad \text{and let} \quad B(x, y) = f(x, y) - f(x, b) \quad \text{for} \ (x, y) \in A. \]
By the Mean Value Theorem,
\[ S(x, y) = A(x, y) - A(x, b) = \frac{\partial A}{\partial y}(x, \eta_A)(y - b) \]
\[ = \frac{\partial f}{\partial y}(x, \eta_A) - \frac{\partial f}{\partial y}(a, \eta_A) \]
\[ = \frac{\partial^2 f}{\partial x \partial y}(\xi_A, \eta_A)(x - a)(y - b) \]
for some $\eta_A$ strictly between $b$ and $y$ and some $\xi_A$ strictly between $a$ and $x$; thus
\[ \lim_{(x, y) \to (a, b)} \frac{S(x, y)}{(x - a)(y - b)} = \frac{\partial^2 f}{\partial x \partial y}(a, b). \]
Again by the Mean Value Theorem,
\[ S(x, y) = B(x, y) - B(a, y) = \frac{\partial B}{\partial x}(\xi_B, y)(x - a) \]
\[ = \frac{\partial f}{\partial x}(\xi_B, y) - \frac{\partial f}{\partial x}(\xi_B, b) \]
\[ = \frac{\partial^2 f}{\partial y \partial x}(\xi_B, \eta_B)(x - a)(y - b) \]
for some $\xi_B$ strictly between $a$ and $x$ and some $\eta_A$ strictly between $b$ and $y$; thus
\[ \lim_{(x, y) \to (a, b)} \frac{S(x, y)}{(x - a)(y - b)} = \frac{\partial^2 f}{\partial y \partial x}(a, b). \]
\[ \square \]

**Remark 1.1.** It turns out there is at least two more versions of this Theorem with different hypotheses but the same conclusion. They each have their merits.