1. The equality of mixed partial derivatives.

Theorem 1.1. Suppose $A \subset \mathbf{R}^2$ and

$$f : A \to \mathbf{R}.$$

Suppose (a, b) is an interior point of A near which the partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

exist. Suppose, in addition, that

$$\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$$

exist near (a, b) and are continuous at (a, b). Then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Proof. Let

$$S(x, y) = f(x, y) - f(x, b) - f(a, y) + f(a, b) \quad \text{for } (x, y) \in A.$$

Let

$$A(x, y) = f(x, y) - f(a, y) \quad \text{and let} \quad B(x, y) = f(x, y) - f(x, b) \quad \text{for } (x, y) \in A.$$

By the Mean Value Theorem,

$$\begin{aligned} S(x, y) = A(x, y) - A(x, b) &= \frac{\partial A}{\partial y}(x, \eta_A)(y - b) \\ &= \frac{\partial f}{\partial y}(x, \eta_A) - \frac{\partial f}{\partial y}(a, \eta_A) \\ &= \frac{\partial^2 f}{\partial x \partial y}(\xi_A, \eta_A)(x - a)(y - b) \end{aligned}$$

for some η_A strictly between b and y and some ξ_A strictly between a and x; thus

$$\lim_{(x,y)\to(a,b)} \frac{S(x, y)}{(x - a)(y - b)} = \frac{\partial^2 f}{\partial x \partial y}(a, b).$$

Again by the Mean Value Theorem,

$$\begin{aligned} S(x, y) = B(x, y) - B(a, y) &= \frac{\partial B}{\partial x}(\xi_B, y)(x - a) \\ &= \frac{\partial f}{\partial x}(\xi_B, y) - \frac{\partial f}{\partial x}(\xi_B, b) \\ &= \frac{\partial^2 f}{\partial y \partial x}(\xi_B, \eta_B)(x - a)(y - b) \end{aligned}$$

for some ξ_B strictly between a and x and some η_A strictly between b and y; thus

$$\lim_{(x,y)\to(a,b)} \frac{S(x, y)}{(x - a)(y - b)} = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

□

Remark 1.1. It turns out there is are at least two more versions of this Theorem with different hypotheses but the same conclusion. They each have their merits.