Minima and maxima.

We fix a positive integer n.

Whenever $\mathbf{a} \in \mathbb{R}^n$ and $0 < r < \infty$ we let

$$\mathbf{U}(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| < r\} \quad \text{and} \quad \mathbf{B}(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| \le r\};$$

the first of these sets is called the **open ball with center a and radius** r and the second is called the **closed ball with center a and radius** r.

We now fix a subset A of $\mathbb{R}^n$.

We let

$$\begin{aligned} \mathbf{int}\, A &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{U}(\mathbf{a}, r) \subset A \text{ for some } r \text{ with } 0 < r < \infty\}; \\ \mathbf{cl}\, A &= \{\mathbf{x} \in \mathbb{R}^n : A \cap \mathbf{U}(\mathbf{a}, r) \neq \emptyset \text{ whenever } 0 < r < \infty\}; \\ \mathbf{bdry}\, A &= \mathbf{cl}\, A \cap \mathbf{cl}\, (\mathbb{R}^n \sim A); \end{aligned}$$

these sets are called the **interior, closure** and **boundary of** A, respectively.

We now suppose that

$$f : A \to \mathbb{R} \quad \text{and} \quad \mathbf{a} \in A.$$

Definition 0.1. We say **a is (global) minimum(maximum) for** f **(on** A**)** if

$$f(\mathbf{a}) \le (\ge) f(\mathbf{x}) \quad \text{whenever } \mathbf{x} \in A.$$

We say **a is a local minimum(maximum) for** f **(on** A**)** if there is $r > 0$ such that

$$f(\mathbf{a}) \le (\ge) f(\mathbf{x}) \quad \text{whenever } \mathbf{x} \in A \cap \mathbf{U}(\mathbf{a}, r).$$

The value $f(\mathbf{a})$ of f at a minimum(maximum) of f is called **the minimum(maximum) value of** f.

Theorem 0.1. Suppose A is closed and bounded and f is continuous. Then f has a minimum and a maximum.

Proof. For each $c \in \mathbf{rng}\, f$ let $F_c = \{x \in A : f(\mathbf{x}) \le c\}$ and note that $F_c \neq \emptyset$ and, because f is continuous, F_c is closed. Because A is closed and bounded the set

$$\bigcap_{c \in \mathbf{rng}\, f} F_c \neq \emptyset.$$

Any member of this set is obviously a minimum for f.

To show a maximum for f exists replace $\le$ by $\ge$ int the definition of F_c. □

Theorem 0.2. Suppose

(i) $\mathbf{a} \in \mathbf{int}\, A$;
(ii) $\mathbf{a}$ is a local maximum or minimum for f;
(iii) $\frac{\partial f}{\partial \mathbf{x}_i}(\mathbf{a})$ exists for each $i \in \{1, \ldots, n\}$.

Then

$$\frac{\partial f}{\partial \mathbf{x}_i}(\mathbf{a}) = 0 \quad \text{for } i \in \{1, \ldots, n\}.$$

Proof. For each $i \in \{1, \dots, n\}$ let $g_i(t) = f(\mathbf{a} + t\mathbf{e}_i)$ for $t \in \mathbb{R}$ such that $\mathbf{a} + t\mathbf{e}_i \in A$; note that 0 is a local maximum or minimum for g_i; and apply the corresponding Theorem from one variable calculus to conclude that

$$\frac{\partial f}{\partial \mathbf{x}_i}(\mathbf{a}) = g_i'(0) = 0.$$

□

Remark 0.1. A point **a** as in the preceding Theorem is called a **critical point for** f.