1. The second derivative test.

Suppose A is an open subset of $\mathbb{R}^2$,

$$f : A \to \mathbb{R},$$

f has continuous second partial derivatives at each point of A and $(a, b) \in A$.

Suppose $(x, y) \in \mathbb{R}^2 \sim \{(a, b)\}$ is such that the line segment joining (a, b) to (x, y) is contained in A. Let

$$x(t) = a + t(x - a) \quad \text{and let} \quad y(t) = b + t(y - b) \quad \text{for } t \in [0, 1].$$

By the Chain Rule we have

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))(x - a) + f_y(x(t), y(t))(y - b)$$

and

$$\begin{aligned}\frac{d^2}{dt^2} f(x(t), y(t)) &= f_{xx}(x(t), y(t))(x - a)^2 + f_{xy}(x(t), y(t))(x - a)(y - b) \\ &+ f_{yx}(x(t), y(t))(y - b)(x - a) + f_{yy}(x(t), y(t))(y - b)^2 \\ &= H(x, y, x - a, y - b)\end{aligned}$$

for any $t \in [0, 1]$, where we have set

$$H(x, y, u, v) = f_{xx}(x(t), y(t))u^2 + 2f_{xy}(x(t), y(t))uv + f_{yy}(x(t), y(t))v^2.$$

If (a, b) is a critical point of f we find using Taylor's Theorem with Lagrange's form of the remainder applied to

$$g(t) = f(x(t), y(t)), \quad t \in [0, 1]$$

that

$$f(x, y) = f(a, b) + H(x(\xi), y(\xi), x - a, y - b) \quad \text{for some } \xi \in (0, 1).$$

It is then a simply matter to show that

(i) if
$$H(a, b, u, v) > 0 \quad \text{for } (u, v) \in \mathbb{R}^2 \sim \{(0, 0)\}$$
then f has a local minimum at (a, b);

(ii) if
$$H(a, b, u, v) < 0 \quad \text{for } (u, v) \in \mathbb{R}^2 \sim \{(0, 0)\}$$
then f has a local maximum at (a, b);

(iii) if
$$H(a, b, u, v) \text{ changes sign for } (u, v) \in \mathbb{R}^2 \sim \{(0, 0)\}$$
then f has neither a local minimum nor a local maximum at (a, b).

Moreover, if we set

$$A = f_{xx}(a, b), \quad B = f_{x,y}(a, b) = f_{yx}(a, b), \quad C = f_{yy}(a, b)$$

then using some elementary linear algebra one may show that (i) is equivalent to

$$AC - B^2 > 0 \quad \text{and} \quad A + C > 0,$$

that (ii) is equivalent to

$$AC - B^2 > 0 \quad \text{and} \quad A + C < 0$$

and that (iii) is equivalent to

$$AC - B^2 < 0.$$

Finally, in (i) and (ii) above, $A + C$ may be replace by either A or C.